CENTRAL EXTENSIONS BY $\mathbf{K}_2$ AND FACTORIZATION LINE BUNDLES

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ABSTRACT. Let X be a smooth, geometrically connected curve over a perfect field k. Given a connected, reductive group G, we prove that central extensions of G by the sheaf $\mathbf{K}_2$ on the big Zariski site of X, studied in Brylinski–Deligne [BD01], are equivalent to factorization line bundles on the Beilinson–Drinfeld affine Grassmannian Gr_G. Our result affirms a conjecture of Gaitsgory-Lysenko [GL16] and classifies factorization line bundles on Gr_G.

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INTRODUCTION

This paper compares two kinds of data parametrizing metaplectic extensions of the Langlands program. One is K-theoretic, and the other has to do with factorization structures on the affine Grassmannian Gr_G.

Let us first explain how these structures arise in the theory.

0.1. K-theoretic metaplectic parameters.

0.1.1. In the classical theory of automorphic forms, one starts with a global field $\mathbf{F}$ and a reductive group G over it. Denote by $\mathbb{A}_{\mathbf{F}}$ the topological ring of adèles of $\mathbf{F}$. The principal objects of interest are certain functions on the homogeneous space $G(\mathbb{A}_{\mathbf{F}})/G(\mathbf{F})$. Roughly speaking, the goal of the Langlands program is to relate them to representations of $\mathrm{Gal}(\overline{\mathbf{F}}/\mathbf{F})$ valued in the L-group of G.

0.1.2. The study of automorphic forms has seen several generalizations, where one replaces $G(\mathbb{A}_{\mathbf{F}})$ by certain topological coverings. The first example of such a covering is the metaplectic group constructed by Weil [We64]. These are double covers of the symplectic groups $\mathrm{Sp}_{2n}(\mathbf{F}_\nu)$, for local fields $\mathbf{F}_\nu$, and combine into a cover of $\mathrm{Sp}_{2n}(\mathbb{A}_{\mathbf{F}})$ equipped with a section over $\mathrm{Sp}_{2n}(\mathbf{F})$.

0.1.3. The existence of interesting topological coverings is by no means restricted to the symplectic group. For any reductive group G, Brylinski–Deligne [BD01] observed that a large class of coverings of $G(\mathbb{A}_{\mathbf{F}})$ arise from K-theoretic data.

To explain their work more precisely, we let $\mathbf{K}_2$ denote the Zariski sheafification of the second algebraic K-theory group. Brylinski–Deligne [BD01] started with a central extension:

$$1 \to \mathbf{K}_2 \to E \to G \to 1 \tag{0.1}$$

of sheaves on the big Zariski site of $\mathbf{F}$. Using the Hilbert symbol on local fields $\mathbf{F}_\nu$, they produced topological central extensions $\widetilde{G}$ of $G(\mathbb{A}_\mathbf{F})$ by the group $\mu_\mathbf{F}$ of roots of unity in $\mathbf{F}$. As a consequence of the reciprocity law of the Hilbert symbol, the central extension $\widetilde{G}$ splits over $G(\mathbf{F})$ [BD01, §10]. This splitting makes it possible to define "metaplectic" automorphic forms as functions on $\widetilde{G}/G(\mathbf{F})$, equivariant against a character of $\mu_\mathbf{F}$ and satisfying certain analytic properties.

0.1.4. The main theorem of *loc.cit.* is that the groupoid of central extensions (0.1) admits a purely combinatorial description. Among other things, the Brylinski–Deligne theorem allows one to define the L-group associated to such a central extension, as has been done by Weissman [We15]. These works bring the study of metaplectic automorphic forms into the broader scope of the Langlands program.

It is thus reasonable to view central extensions by $\mathbf{K}_2$ as metaplectic parameters of the Langlands program and the resulting topological coverings $\widetilde{G}$ as "metaplectic groups" for $G(\mathbb{A}_\mathbf{F})$.

0.2. **Geometric metaplectic parameters.**

0.2.1. Let us now specialize to the function field case, where a more geometric perspective in generalizing the Langlands program is available.

We fix a finite ground field k and a smooth, proper, geometrically connected curve X over k. The letter $\mathbf{F}$ will stand for the field of fractions of X. For simplicity, let us also assume that the reductive group G is defined over k. In the function field setting, automorphic functions can be accessed via ℓ-adic sheaves on the moduli stack Bun_G of principal G-bundles on X (or certain variants thereof in the ramified situation).

0.2.2. The moduli stack Bun_G has a local avatar, known as the affine Grassmannian Gr_G. It is an ind-scheme attached to each closed point x of X and comes equipped with a canonical map to Bun_G. Unlike Bun_G, the affine Grassmannian is not in general (ind-)smooth (e.g. when G is a torus). The usual ind-scheme presentation of the affine Grassmannian is as a colimit of Schubert varieties, whose singularities have representation-theoretic meaning.

Following Beilinson–Drinfeld [BD04], one may also view Gr_G as an ind-scheme over the global curve X. As such, it possesses an additional structure called "factorization." Roughly speaking, factorization structure describes a fusion rule of the fibers of Gr_G as distinct points merge in X. A precise formulation makes use of the Ran space and will be recalled in §0.5.

0.2.3. The ind-scheme Gr_G plays a central role in the Langlands program. Indeed, the category of spherical sheaves Sph_G, which are $\overline{\mathbb{Q}}_\ell$-sheaves on Gr_G equivariant with respect to the arc group (i.e., positive loop group), is equivalent to representations of the Langlands dual group $\check{G}$ under the geometric Satake equivalence of Mirković–Vilonen [MV07]. Their proof uses the factorization structure of Gr_G to construct the symmetry constraint for the convolution monoidal structure on Sph_G.

0.2.4. Motivated by these considerations, Gaitsgory–Lysenko [GL16] proposed to define *geometric metaplectic parameters* as étale gerbes over Gr_G banded by a suitable torsion abelian group $A \subset \overline{\mathbb{Q}}_\ell^\times$, which furthermore respect its factorization structure. These objects are called *factorization gerbes*.

A factorization gerbe allows one to form a "twisted" (or "metaplectic") category of ℓ-adic sheaves on Gr_G. Furthermore, it is possible to replicate the Mirković–Vilonen proof in order to construct the metaplectic geometric Satake equivalence and formulate a vanishing conjecture in the metaplectic geometric Langlands program [GL16, §9-10].

0.3. Relationship between the two.

0.3.1. Let us now turn to the problem addressed in the present paper, which is a comparison of the above two kinds of metaplectic parameters. Since the problem is independent of the global geometry of X, we shall formulate it for any smooth, geometrically connected curve X (i.e., *not* necessarily proper) over a perfect ground field k.

We shall consider the groupoid of central extensions of G by $\mathbf{K}_2$, over the big Zariski site of X rather than its field of fractions[1]. We denote this groupoid by $\mathbf{CExt}(G, \mathbf{K}_2)$.

0.3.2. Subject to a restriction on $\mathrm{char}(k)$, Gaitsgory [Ga18] defined a functor from the groupoid $\mathbf{CExt}(G, \mathbf{K}_2)$ to the (2-)category $\mathbf{Ge}_A^{\mathrm{fact}}(\mathrm{Gr}_G)$ of factorization gerbes on Gr_G. It is a composition of two functors:

$$\begin{aligned}\mathbf{CExt}(G, \mathbf{K}_2) &\xrightarrow{\Phi_G} \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_G) \\ &\xrightarrow{\mathrm{Kum.}} \mathbf{Ge}_A^{\mathrm{fact}}(\mathrm{Gr}_G).\end{aligned}$$

Here, $\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_G)$ denotes the groupoid of factorization line bundles, i.e., line bundles on Gr_G which respect its factorization structure. The second functor is a standard construction using the Kummer exact sequence (where we fix an element in $A(-1)$.) The first functor Φ_G, a kind of residue map on algebraic K-theory, is more interesting. To wit, it relates K-theoretic data to purely geometric objects. The comparison of the two kinds of metaplectic parameters thus amounts to understanding the behavior of Φ_G.

The restriction on $\mathrm{char}(k)$ enters in the definition of Φ_G—it states that $\mathrm{char}(k)$ cannot divide a certain integer N_G which depends on G. The integer N_G is the index of the subgroup of the group of Weyl-invariant, integral quadratic forms on the co-weight lattice generated by Chern classes of representations $G \to \mathrm{GL}(V)$ (see [Ga18, §0.1.8]). Tautologically, the condition $\mathrm{char}(k) \nmid N_G$ is satisfied when G is a product of general linear groups (or when $\mathrm{char}(k) = 0$).

0.3.3. *Main result.* We can now state our main result, which asserts that Φ_G is an equivalence of categories whenever it is defined. It will appear as Theorem 3.1 in the main text.

Theorem A. *Suppose k is a perfect field, X is a smooth, geometrically connected curve and G is a connected reductive group over k. If $\mathrm{char}(k) \nmid N_G$, then Φ_G is an equivalence:*

$$\Phi_G : \mathbf{CExt}(G, \mathbf{K}_2) \xrightarrow{\sim} \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_G).$$

This result affirms [GL16, Conjecture 3.4.2]. Roughly speaking, it means that no information is lost when we pass from K-theoretic metaplectic data to geometry of the affine Grassmannian.

0.3.4. Let us note some consequences of Theorem A. First, the classification theorem of Brylinski–Deligne [BD01] applies to any regular scheme of finite type over a field. In particular, $\mathbf{CExt}(G, \mathbf{K}_2)$ is equivalent to a groupoid of combinatorial gadgets, to be denoted by $\theta_G(\Lambda_T)$.

We shall establish a commutative triangle (appearing as (2.27) in the main text):

$$\begin{tikzcd} \mathbf{CExt}(G, \mathbf{K}_2) \arrow[rr, "\Phi_G"] \arrow[dr, "\Phi_{\mathrm{BD}}"'] & & \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_G) \arrow[dl, "\Psi"] \\ & \theta_G(\Lambda_T) & \end{tikzcd} \tag{0.2}$$

[1] Any central extension over $\mathbf{F}$ extends to one over X_1 for some open $X_1 \subset X$. Any two such extensions to X_1 become canonically isomorphic over some open $X_2 \subset X_1$.

where the functor Ψ is defined in explicit terms (i.e., without recourse to algebraic K-theory). Therefore, Theorem A implies a combinatorial classification of factorization line bundles on Gr_G. The notation $\theta_G(\Lambda_T)$ is meant to recall the groupoid of "θ-data" considered by Beilinson–Drinfeld [BD04], whose classification of factorization line bundles on the space of colored divisors is a precursor to our theorem.

0.3.5. Another application of our theorem is the following.

Corollary B. *Suppose we are under the hypothesis of Theorem A and X is furthermore proper. Then every factorization line bundle on Gr_G canonically descends to Bun_G.*

Indeed, this follows from the fact that the composition:

$$\mathbf{CExt}(G, \mathbf{K}_2) \xrightarrow{\Phi_G} \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_G) \to \mathbf{Pic}(\mathrm{Gr}_G)$$

factors through $\mathbf{Pic}(\mathrm{Bun}_G)$ (see [Ga18, §2.4]). Our Corollary may be viewed as an analogue of Gaitsgory's theorem [Ga13] on cohomological contractibility of the fibers of $\mathrm{Gr}_G \to \mathrm{Bun}_G$.

0.4. Our strategy.

0.4.1. We should mention first that our proof of Theorem A relies on the classification theorem of Brylinski–Deligne, a fact which has two practical implications:

(a) One does *not* need to know the precise definition of Φ_G in order to understand our proof; in fact, as long as Φ_G gives the correct value on regular test schemes $S \to \mathrm{Gr}_G$ (where it is defined using Gersten's resolution of $\mathbf{K}_2$) and satisfies some reasonable properties, then our proof runs through.
(b) After all functors in the triangle (0.2) are defined, checking that it commutes is an essential step towards the proof, and takes up a large part of our work.

A proof of Theorem A without using the Brylinski–Deligne classification would certainly be desirable, but the authors could not find one.[2]

0.4.2. Assuming the commutativity of (0.2) (which will be proved in §2), our proof of the main theorem proceeds by checking that Ψ is an equivalence for various kinds of reductive groups G. We summarize the key ideas and make attributions below (although the main text is organized somewhat differently):

Step 1: $G = T$ is a (split) torus. This case amounts to showing that $\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_T)$ is equivalent to θ-data for the lattice Λ_T. This is the content of §1. In fact, we will show that the same is true for factorization line bundles on various versions of Gr_T. This part of the proof relies on A. Beilinson and V. Drinfeld's classification of factorization line bundles on Λ_T-colored divisors of X (see [BD04]) and the Pic-contractibility of $\mathrm{Ran}(X)$ ([Ta19]).

Step 2: G is semisimple and simply connected. This case is essentially reduced to classifying line bundles on Gr_G at a point of the curve X, and the latter has been worked out by G. Faltings [Fa03]. Since this case is also needed in proving the commutativity of (0.2), it will appear along with it in §2.

Step 3: The derived subgroup G_{der} is simply connected. This case essentially follows from the two previous ones. More precisely, let T_1 be the torus G/G_{der}. We observe that Gr_G is an étale-locally trivial fiber bundle over Gr_{T_1}, with typical fiber $\mathrm{Gr}_{G_{\mathrm{der}}}$. We then use our knowledge from Step 2 to study when a factorization line bundle on Gr_G descends to Gr_{T_1}, and we use Step 1 to classify the ones that are pulled back from the base.

[2] As of now, even the definition of Φ_G appeals to the Brylinski–Deligne classification ([Ga18, §5.1]).

Step 4: An arbitrary reductive group G. This follows from the previous cases, by h-descent of line bundles on derived schemes.[3] Steps 3 and 4 form the content of §3.

0.5. Notations and conventions.

0.5.1. Unlike the main references [GL16] [Ga18], we do *not* need the theory of ∞-categories. Hence terms such as categories, groupoids, prestacks, etc., are understood in the classical sense.

Moreover, the prestacks we consider in the present paper are 0-truncated. Namely, they are synonymous to presheaves on the category of affine schemes. However, in order to stay consistent with existing literature, we shall continue to call them prestacks.

0.5.2. Throughout the paper, we let k be an *algebraically closed* field. The general case of a perfect field is handled using Galois descent. The fact that central extensions by $\mathbf{K}_2$ satisfy Galois descent follows from work of Colliot-Thélène and Suslin. We refer the reader to [BD01, §2] for a detailed discussion.

0.5.3. We let X be a connected, smooth algebraic curve over k.

0.5.4. Let $\mathrm{Ran}(X)$ denote the Ran space associated to X, regarded as a prestack (in fact, a presheaf). For an affine test scheme S over k, an element of $\mathrm{Maps}(S, \mathrm{Ran}(X))$ is by definition a finite subset $x^I = \{x^{(1)}, \cdots, x^{(|I|)}\}$ of $\mathrm{Maps}(S, X)$.

The prestack $\mathrm{Ran}(X)$ has an explicit presentation as a colimit of schemes. Let $\mathbf{fSet}^{\mathrm{surj}}$ denote the category of finite nonempty sets with surjections as morphisms. Then we have an equivalence:

$$\mathrm{Ran}(X) \xrightarrow{\sim} \operatorname*{colim}_{I \in \mathbf{fSet}^{\mathrm{surj}}} X^I,$$

where for each $I_1 \twoheadrightarrow I_2$, the corresponding map $X^{I_2} \to X^{I_1}$ is the diagonal embedding. We refer the reader to [Ga13, §1] for basic properties of the Ran space.

0.5.5. For a finite nonempty set I, we let $\mathrm{Ran}(X)^{\times I}_{\mathrm{disj}}$ denote the open locus in $\mathrm{Ran}(X)^{\times I}$ where the sets of points associated to distinct elements $i_1 \neq i_2 \in I$ are pairwise disjoint.

A prestack $\mathcal{Y}$ over $\mathrm{Ran}(X)$ is a *factorization prestack* if its pullback $\sqcup^* \mathcal{Y}$ along the map of taking disjoint union:

$$\sqcup : \mathrm{Ran}(X)^{\times I}_{\mathrm{disj}} \to \mathrm{Ran}(X)$$

comes equipped with an identification with the restriction $\mathcal{Y}^{\times I}|_{\mathrm{Ran}(X)^{\times I}_{\mathrm{disj}}}$ for each I. This identification is required to satisfy the obvious compatibility condition for compositions along surjections of finite nonempty sets $I_1 \twoheadrightarrow I_2$.

0.5.6. Let $\mathcal{Y}$ be a factorization prestack over $\mathrm{Ran}(X)$. A *factorization line bundle* on $\mathcal{Y}$ is a line bundle $\mathcal{L}$ together with an isomorphism

$$\sqcup^* \mathcal{L} \xrightarrow{\sim} \mathcal{L}^{\boxtimes I}|_{\mathcal{Y}^{\times I}|_{\mathrm{Ran}(X)^{\times I}_{\mathrm{disj}}}} \tag{0.3}$$

over the factorization isomorphism $\sqcup^* \mathcal{Y} \xrightarrow{\sim} \mathcal{Y}^{\times I}|_{\mathrm{Ran}(X)^{\times I}_{\mathrm{disj}}}$, satisfying compatibility for compositions. In fact, it suffices to specify isomorphisms (0.3) for $|I| = 2$, and check the compatilibity conditions for $|I| \leq 3$.

[3] Aside from this descent technique, which was suggested to us by D. Gaitsgory, our paper lives entirely within classical (i.e., non-derived) algebraic geometry.

0.5.7. Let G be a connected, reductive group over k. We write G_{der} for the derived subgroup of G, and $\widetilde{G}_{\mathrm{der}}$ for its universal cover.

When we have fixed a maximal torus $T \subset G$, the notations T_{der} and $\widetilde{T}_{\mathrm{der}}$ will be used to denote the induced maximal tori in G_{der} and $\widetilde{G}_{\mathrm{der}}$.

0.5.8. We write Gr_G for the Beilinson–Drinfeld affine Grassmannian associated to G. For a test affine scheme S, the set $\mathrm{Maps}(S, \mathrm{Gr}_G)$ consists of triples $(\{x^I\}, \mathcal{P}_G, \alpha)$, where:

(a) x^I is a finite subset of $\mathrm{Maps}(S, X)$;
(b) $\mathcal{P}_G$ is a(n étale-locally trivial) G-bundle over $S \times X$;
(c) α is a trivialization of $\mathcal{P}_G$ over $S \times X - \bigcup_{i\in I} \Gamma_{x^{(i)}}$, where $\Gamma_{x^{(i)}}$ denotes the graph of $x^{(i)}$.

The morphism $\mathrm{Gr}_G \to \mathrm{Ran}(X)$ is ind-schematic and of ind-finite type, and realizes Gr_G as a factorization prestack over $\mathrm{Ran}(X)$. The base change of Gr_G along $X^I \to \mathrm{Ran}(X)$ will be denoted by Gr_{G,X^I}. We refer the reader to [Zh16] for properties of Gr_G.

0.5.9. We let $\mathcal{L}^+G$ (resp. $\mathcal{L}G$) denote the arc (resp. loop) group, regarded as factorization group prestacks over $\mathrm{Ran}(X)$. For a test affine scheme S, a lift of $x^I : S \to \mathrm{Ran}(X)$ to $\mathcal{L}^+G$ (resp. $\mathcal{L}G$) is given by a map from the formal completion D_{x^I} (resp. punctured formal completion $\mathring{D}_{x^I} := D_{x^I} \setminus \bigcup_{i\in I} \Gamma_{x^{(i)}}$) of $\bigcup_{i\in I} \Gamma_{x^{(i)}}$ inside $S \times X$ to G.

Furthermore, the projection $\mathcal{L}^+G \to \mathrm{Ran}(X)$ is schematic (but not of finite type) and $\mathcal{L}G \to \mathrm{Ran}(X)$ is ind-schematic. The affine Grassmannian Gr_G can be expressed as the quotient $\mathcal{L}G/\mathcal{L}^+G$ of étale sheaves.

0.5.10. For a closed point $x \in X$, we denote by $\mathcal{O}_x$ the *completed* local ring at x and $\mathcal{K}_x$ its localization at a uniformizer. The fibers of the above prestacks at a closed point $x \in X$ will be denoted by $\mathrm{Gr}_{G,x}$, $\mathcal{L}_xG$, and $\mathcal{L}_x^+G$. Thus $\mathcal{L}_xG(k) \cong G(\mathcal{K}_x)$ and $\mathcal{L}_x^+G(k) \cong G(\mathcal{O}_x)$.

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1. Factorization line bundles for tori

In this section, we prove that factorization line bundles on various versions of Gr_T (e.g., combinatorial, rational) are all classified by θ-data.

1.1. The many faces of Gr_T.

1.1.1. Suppose T is a torus over k. Let Λ_T denote its co-character lattice. We will first introduce a few variants of the affine Grassmannian Gr_T. They are summarized in the following commutative diagram:

$$\begin{tikzcd} \mathrm{Gr}_{T,\mathrm{comb}} \arrow[r] & \mathrm{Gr}_T \arrow[r] \arrow[d] & \mathrm{Div}(X) \underset{\mathbb{Z}}{\otimes} \Lambda_T \arrow[d] \\ & \mathrm{Gr}_{T,\mathrm{lax}} \arrow[r] & \mathrm{Gr}_{T,\mathrm{rat}} \end{tikzcd} \tag{1.1}$$

1.1.2. *The combinatorial variant.* Consider an index category whose objects are pairs $(I, \lambda^{(I)})$, where I is a finite set, and $\lambda^{(I)}$ is an I-family of elements in Λ_T (its element corresponding to $i \in I$ is denoted by $\lambda^{(i)}$). A morphism $(I, \lambda^{(I)}) \to (J, \lambda^{(J)})$ in this category consists of a *surjective* map $\varphi : I \twoheadrightarrow J$ such that $\lambda^{(j)} = \sum_{i \in \varphi^{-1}(j)} \lambda^{(i)}$ for all $j \in J$. We set:

$$\mathrm{Gr}_{T,\mathrm{comb}} := \underset{(I,\lambda^{(I)})}{\mathrm{colim}}\, X^I.$$

$\mathrm{Gr}_{T,\mathrm{comb}}$ is a factorization prestack over $\mathrm{Ran}(X)$. Furthermore, we have a canonical map $\mathrm{Gr}_{T,\mathrm{comb}} \to \mathrm{Gr}_T$ sending an S-point $x^I : S \to X^I$ corresponding to $(I, \lambda^{(I)})$ to the triple $(\{x^{(i)}\}, \bigotimes_{i\in I} \mathcal{O}(\lambda^{(i)}\Gamma_{x^i}), \alpha)$ where α is the tautological trivialization.

1.1.3. *The lax variant.* We let $\mathrm{Gr}_{T,\mathrm{lax}}$ denote the lax prestack[4] whose value at S is the *category* whose objects are triples $(x^I, \mathcal{P}_T, \alpha)$ as in $\mathrm{Gr}_T(S)$, but there is a morphism:

$$(x^I, \mathcal{P}_T, \alpha) \to (x^J, \mathcal{P}'_T, \alpha'),$$

whenever $x^I \subset x^J$, $\mathcal{P}_T \xrightarrow{\sim} \mathcal{P}'_T$, and the trivialization α restricts to α' over the complement of $\bigcup_{j\in J} \Gamma_{x^{(j)}}$. Such a morphism is non-invertible when $x^I \subset x^J$ is a proper inclusion.

$\mathrm{Gr}_{T,\mathrm{lax}}$ is a factorization lax prestack over the lax version of the Ran space $\mathrm{Ran}(X)_{\mathrm{lax}}$. Furthermore, we have a canonical map $\mathrm{Gr}_T \to \mathrm{Gr}_{T,\mathrm{lax}}$ sending $(x^I, \mathcal{P}_T, \alpha)$ to the very same object.

1.1.4. *The rational variant.* We define $\mathrm{Gr}_{T,\mathrm{rat}}$ as a prestack whose value at S is the groupoid of T-bundles $\mathcal{P}_T$ over $S \times X$ equipped with a *rational trivialization*, i.e., for some open $U \subset S \times X$ which is schematically dense after arbitrary base change $S' \to S$, the T-bundle $\mathcal{P}_T$ admits a trivialization over U; we regard two rational trivializations as equivalent if they agree on the overlaps.

Even though $\mathrm{Gr}_{T,\mathrm{rat}}$ does not live over any version of the Ran space, one can still make sense of factorization line bundles (or any other gadget) over $\mathrm{Gr}_{T,\mathrm{rat}}$. Namely, it is a line bundle $\mathcal{L}$ over $\mathrm{Gr}_{T,\mathrm{rat}}$ together with isomorphisms:

$$c_{\mathcal{P}_T^{(1)},\mathcal{P}_T^{(2)}} : \mathcal{L}\big|_{\mathcal{P}_T} \xrightarrow{\sim} \mathcal{L}\big|_{\mathcal{P}_T^{(1)}} \otimes \mathcal{L}\big|_{\mathcal{P}_T^{(2)}},$$

whenever $\mathcal{P}_T^{(1)}$ (resp. $\mathcal{P}_T^{(2)}$) admits a trivialization over $U^{(1)}$ (resp. $U^{(2)}$) such that the complements of $U^{(1)}$ and $U^{(2)}$ are disjoint, and $\mathcal{P}_T$ is the gluing of $\mathcal{P}_T^{(1)}\big|_{U^{(2)}}$ and $\mathcal{P}_T^{(2)}\big|_{U^{(1)}}$ along $U^{(1)} \cap U^{(2)}$, where they are both trivialized. The isomorphisms $c_{\mathcal{P}_T^{(1)},\mathcal{P}_T^{(2)}}$ are required to satisfy the obvious compatibility conditions in the presence of three T-bundles.

Remark 1.1. The objects $\mathrm{Gr}_{T,\mathrm{lax}}$ and $\mathrm{Gr}_{T,\mathrm{rat}}$ have analogues for a general group G, but we will not use them in this paper.

1.1.5. *Colored divisors.* Recall the prestack $\mathrm{Div}(X)$ whose value at S is the abelian group of Cartier divisors of $S \times X$ relative to S. We take $\mathrm{Div}(X) \underset{\mathbb{Z}}{\otimes} \Lambda_T$ as its extension of scalars to Λ_T. There is a morphism $\mathrm{Div}(X) \to \mathrm{Gr}_{\mathbb{G}_m,\mathrm{rat}}$ defined by associating to a Cartier divisor D the line bundle $\mathcal{O}_{S\times X}(D)$. It extends to a morphism $\mathrm{Div}(X) \underset{\mathbb{Z}}{\otimes} \Lambda_T \to \mathrm{Gr}_{T,\mathrm{rat}}$.

As in the previous case, we make sense of factorization line bundles over $\mathrm{Div}(X) \underset{\mathbb{Z}}{\otimes} \Lambda_T$ as follows. It is a line bundle $\mathcal{L}$ together with isomorphisms:

$$c_{D_1,D_2} : \mathcal{L}\big|_{D_1+D_2} \xrightarrow{\sim} \mathcal{L}\big|_{D_1} \otimes \mathcal{L}\big|_{D_2},$$

[4]See [Ga15, §2] for an introduction to lax prestacks.

whenever the support of D_1 and D_2 are disjoint. The isomorphisms c_{D_1,D_2} are required to satisfy the obvious compatibility conditions for three divisors.

1.2. Classification statements.

1.2.1. *θ-data.* We recall the notion of θ-data for a lattice Λ due to Beilinson–Drinfeld [BD04, §3.10.3]. The Picard groupoid $\theta(\Lambda)$ consists of triples $(q, \mathcal{L}^{(\lambda)}, c_{\lambda,\mu})$ where:

(a) $q \in Q(\Lambda, \mathbb{Z})$ is an integral valued quadratic form on Λ; we use κ to denote its symmetric bilinear form, defined by the formula: $\kappa(\lambda, \mu) := q(\lambda + \mu) - q(\lambda) - q(\mu)$;
(b) $\mathcal{L}^{(\lambda)}$ is a system of line bundles on X parametrized by $\lambda \in \Lambda$, and
(c) $c_{\lambda,\mu}$ are isomorphisms:

$$c_{\lambda,\mu} : \mathcal{L}^{(\lambda)} \otimes \mathcal{L}^{(\mu)} \xrightarrow{\sim} \mathcal{L}^{(\lambda+\mu)} \otimes \omega_X^{\kappa(\lambda,\mu)}, \tag{1.2}$$

which are associative, and satisfy a κ-*twisted* commutativity condition, i.e.

$$c_{\lambda,\mu}(a \otimes b) = (-1)^{\kappa(\lambda,\mu)} \cdot c_{\mu,\lambda}(b \otimes a). \tag{1.3}$$

Remark 1.2. The authors of [BD04] work in the setting of $\mathbb{Z}/2\mathbb{Z}$-graded line bundles, so what we call θ-data corresponds to what they call *even* θ-data.

1.2.2. *Shifted θ-data.* For later purposes, we also introduce a Picard groupoid $\theta^+(\Lambda)$ consisting of triples $(q, \mathcal{L}^{(\lambda)}, c^+_{\lambda,\mu})$, where we replace (1.2) by isomorphisms $c^+_{\lambda,\mu} : \mathcal{L}^{(\lambda)} \otimes \mathcal{L}^{(\mu)} \xrightarrow{\sim} \mathcal{L}^{(\lambda+\mu)}$ and also demand that they are associative and satisfy the κ-twisted commutativity condition. Clearly, we have an equivalence:

$$\theta(\Lambda) \xrightarrow{\sim} \theta^+(\Lambda), \quad (q, \mathcal{L}^{(\lambda)}) \rightsquigarrow (q, \mathcal{L}^{(\lambda)} \otimes \omega_X^{q(\lambda)}).$$

Lemma 1.3. *There is a canonical equivalence of Picard groupoids* $\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{T,\mathrm{comb}}) \xrightarrow{\sim} \theta(\Lambda_T)$.

Proof. Given a factorization line bundle over $\mathrm{Gr}_{T,\mathrm{comb}}$, we denote its pullback along the inclusion $X \to \mathrm{Gr}_{T,\mathrm{comb}}$ corresponding to $(\{1\}, \lambda)$ by $\mathcal{L}^{(\lambda)}$, and its pullback along $X^2 \to \mathrm{Gr}_{T,\mathrm{comb}}$ corresponding to $(\{1,2\}, (\lambda,\mu))$ by $\mathcal{L}^{(\lambda,\mu)}$. The factorization isomorphism shows that there is an isomorphism $\mathcal{L}^{(\lambda)} \boxtimes \mathcal{L}^{(\mu)}\big|_{x^2-\Delta} \xrightarrow{\sim} \mathcal{L}^{(\lambda,\mu)}$. It extends to an isomorphism

$$\mathcal{L}^{(\lambda)} \boxtimes \mathcal{L}^{(\mu)} \xrightarrow{\sim} \mathcal{L}^{(\lambda,\mu)} \otimes \mathcal{O}_{X^2}(-\kappa(\lambda,\mu)\Delta), \tag{1.4}$$

for some uniquely determined integer $\kappa(\lambda, \mu)$; its dependency on λ, μ is bilinear, by considering $\mathcal{L}^{(\lambda,\mu,\nu)}$ for a triple $(\{1,2,3\}, (\lambda,\mu,\nu))$, using the compatibility between factorization isomorphism and composition. Since $\mathcal{L}^{(\lambda,\mu)}$ restricts to $\mathcal{L}^{(\lambda+\mu)}$ along $\Delta \hookrightarrow X^2$, the isomorphism (1.4) restricts to a system of isomorphisms $c_{\lambda,\mu}$ as in (1.2).

Next, because the factorization isomorphisms are Σ_2-invariant, so are the isomorphisms (1.4). In other words, we have a commutative diagram:

$$\begin{tikzcd}
\mathcal{L}^{(\lambda)} \boxtimes \mathcal{L}^{(\mu)} \arrow[r, "\sim"] \arrow[d, "\cong"] & \mathcal{L}^{(\lambda,\mu)} \otimes \mathcal{O}_{X^2}(-\kappa(\lambda,\mu)\Delta) \arrow[d, "\cong"] \\
\sigma^*(\mathcal{L}^{(\mu)} \boxtimes \mathcal{L}^{(\lambda)}) \arrow[r, "\sim"] & \sigma^*\mathcal{L}^{(\mu,\lambda)} \otimes \sigma^*\mathcal{O}_{X^2}(-\kappa(\mu,\lambda)\Delta),
\end{tikzcd} \tag{1.5}$$

where σ is the isomorphism $X^{(\lambda,\mu)} \xrightarrow{\sim} X^{(\mu,\lambda)}$. One deduces from this fact that κ is also symmetric. Restricting (1.5) to the diagonal, we obtain a commutative diagram:

$$\begin{tikzcd}
\mathcal{L}^{(\lambda)} \otimes \mathcal{L}^{(\mu)} \arrow[r, "c_{\lambda,\mu}"] \arrow[d, "\cong"] & \mathcal{L}^{(\lambda+\mu)} \otimes \omega_X^{\kappa(\lambda,\mu)} \arrow[d, "(-1)^{\kappa(\lambda,\mu)}"] \\
\mathcal{L}^{(\mu)} \otimes \mathcal{L}^{(\lambda)} \arrow[r, "c_{\mu,\lambda}"] & \mathcal{L}^{(\mu+\lambda)} \otimes \omega_X^{\kappa(\mu,\lambda)}
\end{tikzcd}$$

where the multiplication by $(-1)^{\kappa(\lambda,\mu)}$ appears because the isomorphism $\mathcal{O}_{X^2}(-\Delta)\big|_{\Delta} \xrightarrow{\sim} \omega_X$ is only Σ_2-invariant *up to a sign*. This commutative diagram expresses the identity (1.3). Finally taking $\lambda = \mu$, we see that $(-1)^{\kappa(\lambda,\lambda)} = 1$, so $\kappa(\lambda,\lambda) = 2q(\lambda)$ for a uniquely determined integer $q(\lambda)$. Thus we have define an integral quadratic form q on Λ_T.

The above procedure defines the functor $\mathbf{Pic}^{\text{fact}}(\mathrm{Gr}_{T,\text{comb}}) \to \Theta(\Lambda_T; \mathbf{Pic})$. Checking that it is an equivalence is straightforward. $\square$

1.2.3. We can now state the main result of this section. By pulling back along the morphisms of (1.1), we obtain a diagram of Picard groupoids, where the leftmost equivalence comes from Lemma 1.3:

$$\begin{tikzcd}
\theta(\Lambda_T) & \mathbf{Pic}^{\text{fact}}(\mathrm{Gr}_{T,\text{comb}}) \arrow[l, "\sim"'] & \mathbf{Pic}^{\text{fact}}(\mathrm{Gr}_T) \arrow[l] & \mathbf{Pic}^{\text{fact}}(\mathrm{Div}(X) \underset{\mathbb{Z}}{\otimes} \Lambda_T) \arrow[l, "(a)"'] \\
& & \mathbf{Pic}^{\text{fact}}(\mathrm{Gr}_{T,\text{lax}}) \arrow[u, "(c)"] & \mathbf{Pic}^{\text{fact}}(\mathrm{Gr}_{T,\text{rat}}) \arrow[l, "(b)"'] \arrow[u]
\end{tikzcd} \tag{1.6}$$

Proposition 1.4. *All morphisms in (1.6) are equivalences.*

Proof. We shall deduce from existing literature how each of the labeled maps is an equivalence:

(a) By [BD04, §3.10.7, Proposition], the composition of the top row defines an equivalence: $\mathbf{Pic}^{\text{fact}}(\mathrm{Div}(X) \underset{\mathbb{Z}}{\otimes} \Lambda_T) \xrightarrow{\sim} \theta(\Lambda_T)$. This shows that the map (a) has a left inverse.

(b) By [Ba12, Proposition 5.2.2], the map $\mathrm{Gr}_{T,\text{lax}} \to \mathrm{Gr}_{T,\text{rat}}$ induces an equivalence after fppf sheafification. Hence pulling back defines an equivalence $\mathbf{Pic}(\mathrm{Gr}_{T,\text{rat}}) \xrightarrow{\sim} \mathbf{Pic}(\mathrm{Gr}_{T,\text{lax}})$. One immediately checks that the additional data defining factorization structures on both are also equivalent. Hence (b) is an equivalence.

(c) By [Zh16, Theorem 4.3.9(2)], pulling back along $\mathrm{Gr}_T \to \mathrm{Gr}_{T,\text{rat}}$ defines an equivalence on *rigidified* line bundles[5]. On the other hand, every factorization line bundle on Gr_T pulls back to one along the unit section $\mathrm{Ran}(X) \to \mathrm{Gr}_T$, which is canonically trivial by Lemma 1.3 (applied to the trivial group). Thus a factorization line bundle on Gr_T descends to a line bundle on $\mathrm{Gr}_{T,\text{rat}}$, and the result has a canonical factorization structure as well, so we have an equivalence $\mathbf{Pic}^{\text{fact}}(\mathrm{Gr}_{T,\text{rat}}) \xrightarrow{\sim} \mathbf{Pic}^{\text{fact}}(\mathrm{Gr}_T)$. This shows that (c) is an equivalence.

The undecorated maps in (1.6) are now equivalences by the 2-out-of-3 property. $\square$

Remark 1.5. When X is proper, [Ca17, Theorem 2.3.3] shows that the map $\mathrm{Div}(X) \underset{\mathbb{Z}}{\otimes} \Lambda_T \to \mathrm{Gr}_{T,\text{rat}}$ is an isomorphism of prestacks, which immediately implies that factorization line bundles on them are equivalent.

[5][Zh16, Theorem 4.3.9(2)] is not given a proof in *loc.cit.*, and we refer the reader to [Ta19] for a complete proof of the key Pic-contractibility statement involved.

Remark 1.6. We have the following equivalence for any smooth, fiberwise connected, affine group scheme $\mathbf{G}$ over X:

$$\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{\mathbf{G},\mathrm{rat}}) \xrightarrow{\sim} \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{\mathbf{G},\mathrm{lax}}) \xrightarrow{\sim} \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{\mathbf{G}}).$$

This is because the results [Ba12, Proposition 5.2.2] and [Zh16, Theorem 4.3.9(2)] both hold in this general context.

2. Compatibility with the Brylinski–Deligne classification

In this section, we first summarize Brylinski–Deligne's classification of central extensions of G by $\mathbf{K}_2$. Then we construct a functor from $\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_G)$ to the same classification data and we prove that it is compatible with Gaitsgory's functor Φ_G.

2.1. Extensions by $\mathbf{K}_2$.

2.1.1. This subsection serves as a summary of the main result of [BD01]. Let G be a connected, reductive group over k. Fix a maximal torus $T \subset G$. We recall the notations $\theta(\Lambda_T)$ and $\theta^+(\Lambda_T)$ for the θ-data associated to Λ_T (see §1.2.1-1.2.2).

2.1.2. We let $\mathbf{K}_2$ denote the Zariski sheafification of the presheaf on $\mathbf{Sch}^{\mathrm{aff}}_{/X}$ that sends any $S \to X$ to $K_2(S)$. For a connected, reductive group G, we let $\mathbf{CExt}(G, \mathbf{K}_2)$ denote the Picard groupoid of central extensions

$$1 \to \mathbf{K}_2 \to E \to G \to 1, \tag{2.1}$$

in the category of Zariski sheaves of groups on $\mathbf{Sch}^{\mathrm{aff}}_{/X}$. This is Picard groupoid of *Brylinski–Deligne data*.

2.1.3. We will first define a functor

$$\mathbf{CExt}(T, \mathbf{K}_2) \to \theta^+(\Lambda_T). \tag{2.2}$$

Indeed, given a central extension E of T, we construct a triple $(q, \mathcal{L}^{(\lambda)}, c^+_{\lambda,\mu}) \in \theta^+(\Lambda_T)$ from the following procedure:

(a) The commutator in E defines a map $\mathrm{comm} : T \underset{\mathbb{Z}}{\otimes} T \to \mathbf{K}_2$ of Zariski sheaves on $\mathbf{Sch}^{\mathrm{aff}}_{/X}$. For any $\lambda, \mu \in \Lambda_T$, the composition: $\mathbb{G}_m \underset{\mathbb{Z}}{\otimes} \mathbb{G}_m \xrightarrow{\lambda\otimes\mu} T \underset{\mathbb{Z}}{\otimes} T \to \mathbf{K}_2$ is some integral multiple of the universal symbol $\{-,-\}$ (c.f. §3.8 of *loc.cit.*). We call this integer $\kappa(\lambda,\mu)$. One then checks that $\kappa(-,-)$ is the bilinear form associated to some quadratic form q.

(b) Consider the projection $p : \mathbb{G}_m \times X \to X$. Using the vanishing result $\mathrm{R}^1 p_* \mathbf{K}_2 = 0$ of Sherman (c.f. §3.1 of *loc.cit.*), we find an exact sequence of Zariski sheaves on X:

$$1 \to p_*\mathbf{K}_2 \to p_*E \to p_*T \to 1.$$

Pushing out along the symbol map $p_*\mathbf{K}_2 \to \mathbf{K}_1 \cong \mathcal{O}_X^\times$, we obtain a multiplicative $\mathcal{O}_X^\times$-torsor over p_*T. The line bundle $\mathcal{L}^{(\lambda)}$ then arises as the fiber of the section of p_*T defined by $\lambda \in \Lambda_T$.

(c) Note that the aforementioned multiplicative $\mathcal{O}_X$-torsor over p_*T equips the system $\{\mathcal{L}^{(\lambda)}\}$ with the multiplicative structure $c^+_{\lambda,\mu}$. Its failure of commutativity is measured by κ, as desired.

2.1.4. It is proved in *loc.cit.* that (2.2) is an equivalence of Picard groupoids. We record here the *un*shifted version of this equivalence:

$$\mathbf{CExt}(T, \mathbf{K}_2) \xrightarrow{\sim} \theta(\Lambda_T), \tag{2.3}$$

i.e., it is the composition of (2.2) with the equivalence of Picard groupoids $\theta^+(\Lambda_T) \xrightarrow{\sim} \theta(\Lambda_T)$ sending $\mathcal{L}^{(\lambda)}$ to $\mathcal{L}^{(\lambda)} \otimes \omega_X^{-q(\lambda)}$.

2.1.5. We now turn to the general case. Note that there is always a functor:

$$\mathbf{CExt}(G, \mathbf{K}_2) \xrightarrow{\text{res}} \mathbf{CExt}(T, \mathbf{K}_2) \xrightarrow{\sim} \theta(\Lambda_T) \to Q(\Lambda_T, \mathbb{Z}), \tag{2.4}$$

whose image lands in the W-invariant part of $Q(\Lambda_T, \mathbb{Z})$. Thus, we may speak of *the* quadratic form q associated to an extension (2.1).

2.1.6. Suppose G is semisimple and simply connected. Then Theorem 4.7 of *loc.cit.* asserts that (2.4) defines an equivalence: $\mathbf{CExt}(G, \mathbf{K}_2) \xrightarrow{\sim} Q(\Lambda_T, \mathbb{Z})^W$. Thus for a semisimple, simply connected group G, there is a map which associates theta data to a W-invariant quadratic form:

$$Q(\Lambda_T, \mathbb{Z})^W \to \theta(\Lambda_T). \tag{2.5}$$

2.1.7. Let $\widetilde{G}_{\text{der}}$ be the simply connected cover of G_{der}. It contains a maximal torus $\widetilde{T}_{\text{der}}$ which is the preimage of T_{der}. We now let $\theta_G(\Lambda_T)$ denote the Picard groupoid classifying:

(a) a theta datum $(q, \mathcal{L}^{(\lambda)}, c_{\lambda,\mu})$ for Λ_T, where q is Weyl-invariant.
(b) an isomorphism φ between the following theta data for $\Lambda_{\widetilde{T}_{\text{der}}}$:
   - the restriction of $(q, \mathcal{L}^{(\lambda)}, c_{\lambda,\mu})$ to $\Lambda_{\widetilde{T}_{\text{der}}}$;
   - the theta data associated to $q\big|_{\Lambda_{\widetilde{T}_{\text{der}}}}$ via (2.5).

In other words, φ consists of isomorphisms between line bundles, preserving their (ω-twisted) multiplicative structure. We shall call $\theta_G(\Lambda_T)$ the Picard groupoid of *enhanced* theta data. By definition, we have a functor:

$$\Phi_{\text{BD}} : \mathbf{CExt}(G, \mathbf{K}_2) \to \theta_G(\Lambda_T), \tag{2.6}$$

obtained by restrictions to T and $\widetilde{T}_{\text{der}}$. The main theorem of [BD01] is that (2.6) is an equivalence of Picard groupoids, i.e., central extensions of G by $\mathbf{K}_2$ are classified by enhanced theta data.

2.2. Gaitsgory's functor Φ_G.

2.2.1. Under the condition that the characteristic of k does not divide the integer N_G, Gaitsgory [Ga18] constructed a functor:

$$\Phi_G : \mathbf{CExt}(G, \mathbf{K}_2) \to \mathbf{Pic}^{\text{fact}}(\text{Gr}_G). \tag{2.7}$$

Only two features of Φ_G will be used in proving its compatibility with the Brylinski–Deligne classification. We first cast them in informal language:

(a) Given a central extension (2.1), its image under Φ_G is a line bundle $\mathcal{L}$ over Gr_G with additional factorization data; for a *regular* affine scheme $S \to \text{Gr}_G$, we need the restriction $\mathcal{L}\big|_S$ to be given by "taking the residue" along $S \times X \to S$.
(b) Suppose $G = T$ is a torus; we need the functor Φ_T to factor through the Picard groupoid of *multiplicative* factorization line bundles on $\mathcal{L}T$, and for a closed point $x \in X$, we need the multiplicative structure on $\mathcal{L}_x T$ to be given by the "tautological" one.

We will make precise what features (a) and (b) mean in the rest of this subsection, and explain how they can be deduced from *loc.cit.*

2.2.2. Let S be a *regular* affine scheme over k and $\pi : \mathfrak{X} \to S$ be a smooth relative curve, whose fibers are geometrically connected. Furthermore, suppose we have a finite set $\{x^I\}$ of sections $x^{(i)} : S \to \mathfrak{X}$. Let Γ_{x^I} denote the (scheme-theoretic) union of their images, and $U_{x^I} := \mathfrak{X} - \Gamma_{x^I}$ be its complement.

We will construct a functor, referred to hereafter as *taking the residue* along π:

$$\left\{ \begin{matrix} \mathbf{K}_2\text{-gerbes } \mathcal{G} \text{ on } \mathfrak{X} \text{ with} \\ \text{neutralization } \gamma \text{ over } U_{x^I} \end{matrix} \right\} \to \mathbf{Pic}(S). \tag{2.8}$$

Indeed, the datum $(\mathcal{G}, \gamma)$ is equivalent to a section of $\iota^! \mathbf{K}_2[2]$ over $\mathfrak{X}$, where $\iota : \Gamma_{x^I} \hookrightarrow \mathfrak{X}$ is the closed immersion. On the other hand, the Gersten resolution of $\mathbf{K}_2$ on $\mathfrak{X}$ shows that $\iota^! \mathbf{K}_2[2]$ is quasi-isomorphic to the complex concentrated in degrees $[-1, 0]$:

$$\bigoplus_{i \in I} (\iota_{\eta^{(i)}})_* K_1(\eta) \to \bigoplus_{\substack{\mathrm{codim}(\nu)=1 \\ \text{in } \Gamma_{x^I}}} (\iota_\nu)_* \mathbb{Z} \tag{2.9}$$

where $\iota_{\eta^{(i)}}$ (resp. ι_ν) denotes the inclusion of the generic point of the ith section (resp. codimension-one point ν of Γ_{x^I}). On the other hand, $\mathbf{K}_1[1]$ over S is quasi-isomorphic to:

$$(\iota_\eta)_* K_1(\eta) \to \bigoplus_{\substack{\mathrm{codim}(\nu)=1 \\ \text{in } S}} (\iota_\nu)_* \mathbb{Z}.$$

Thus the image of (2.9) under π maps to $\mathbf{K}_1[1]$ via summation. Hence a section of $\iota^! \mathbf{K}_2[2]$ over $\mathfrak{X}$ gives rise to a section of $\mathbf{K}_1[1] \cong \mathcal{O}_S^\times[1]$, i.e., a line bundle on S.

2.2.3. Given an extension E (2.1) and a map $S \to \mathrm{Gr}_G$ specified by the triple $(\{x^I\}, \mathcal{P}_G, \alpha)$ where $\mathcal{P}_G$ is *Zariski* locally trivial, we obtain a (Zariski) $\mathbf{K}_2$-gerbe $\mathcal{G}$ over $S \times X$, which classifies an E-torsor $\mathcal{P}_E$ equipped with an identification of its induced G-torsor $(\mathcal{P}_E)_G \xrightarrow{\sim} \mathcal{P}_G$. The trivialization α gives rise to a neutralization γ of $\mathcal{G}$ over U_{x^I}.

Suppose S is regular, then $(\mathcal{G}, \gamma)$ produces a line bundle on S by taking the residue (2.8) along $\pi : S \times X \to S$. This process also applies when $\mathcal{P}_G$ is only étale locally trivial, since étale locally on S the bundle $\mathcal{P}_G$ becomes Zariski locally trivial (see [DS95]). The fact that $\Phi_G(E)\big|_S$ naturally agrees with this line bundle is the content of [Ga18, §2.3]; this is what we meant in part (a) of §2.2.1.

2.2.4. Recall that a *multiplicative* line bundle $\mathcal{L}$ on $\mathcal{L}G$ amounts to the additional isomorphism:

$$\mathrm{mult}^* \mathcal{L} \xrightarrow{\sim} \mathcal{L} \boxtimes \mathcal{L} \tag{2.10}$$

over $\mathcal{L}G \underset{\mathrm{Ran}(X)}{\times} \mathcal{L}G$ that satisfies the cocycle condition on the triple product. If $\mathcal{L}$ is a factorization line bundle, then being multiplicative amounts to an isomorphism (2.10) that is compatible with the factorization structures on both sides.

We let $\mathbf{Pic}^{\mathrm{fact}, \times}(\mathcal{L}G)$ (resp. $\mathbf{Pic}^{\mathrm{fact}, \times}_{/\mathcal{L}^+G}(\mathcal{L}G)$) denote the Picard groupoid of multiplicative factorization line bundles on $\mathcal{L}G$ (resp. together with a trivialization *as such* over $\mathcal{L}^+G$). Clearly, there is a descent functor:

$$\mathbf{Pic}^{\mathrm{fact}, \times}_{/\mathcal{L}^+G}(\mathcal{L}G) \to \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_G).$$

We now state part (b) of §2.2.1 as a lemma:

Lemma 2.1. *(a) The functor Φ_T factors through $\mathbf{Pic}^{\mathrm{fact}, \times}_{/\mathcal{L}^+T}(\mathcal{L}T)$, i.e., $\Phi_T(E)$ has a canonical multiplicative structure over $\mathcal{L}T$, trivialized over $\mathcal{L}^+T$;*

(b) *Over a closed point $x \in X$, the restriction of the above multiplicative structure to the abstract group $T(\mathcal{K}_x)$*[6] *agrees with that on the $k^\times$-torsor coming from the push-out of*

$$0 \to \mathbf{K}_2(\mathcal{K}_x) \to E(\mathcal{K}_x) \to T(\mathcal{K}_x) \to 0 \tag{2.11}$$

along the residue map $\mathbf{K}_2(\mathcal{K}_x) \to k^\times$. The same holds over any field extension $k \subset k'$.

Remark 2.2. Part (b) makes sense since $\Phi_T(E)\big|_t$ for $t \in T(\mathcal{K}_x)$ agrees with the $k^\times$-torsor induced from (2.11); this follows from the description of $\Phi_T(E)$ on regular test schemes (§2.2.3).

Proof of Lemma 2.1. Recall that $\mathcal{L} := \Phi_T(E)$ is constructed as follows. The datum E can be interpreted as a pointed morphism $e : X \times \mathrm{B}\,T \to \mathrm{B}^2\,\mathbf{K}_2$. Let $\mathbf{K}$ denote the full K-theory spectrum, regarded as a Zariski sheaf on $\mathbf{Sch}^{\mathrm{aff}}$. Then e lifts (non-uniquely) to some $\tilde{e} : X \times \mathrm{B}\,T \to \mathbf{K}_{\geq 2}$ ([Ga18, §5.3.1]). Hence the data $(\{x^I\}, \mathcal{P}_T, \alpha)$ of an S-point of Gr_T (where we may again assume $\mathcal{P}_T$ to be Zariski-locally trivial) give us a section of $\mathbf{K}_{\geq 2}$ over $S \times X$ with support on Γ_{x^I}. The line bundle $\mathcal{L}_{\tilde{e}}\big|_S$ is then constructed using the map:

$$\tau^{\leq 0}\pi_*\iota^!\mathbf{K}_{\geq 2} \to \mathcal{O}_S^\times[1] \tag{2.12}$$

(c.f. (3.2.2) of *loc.cit.*). For two lifts $\tilde{e}$ and $\tilde{e}'$, we need to produce a canonical isomorphism $\mathcal{L}_{\tilde{e}} \xrightarrow{\sim} \mathcal{L}_{\tilde{e}'}$. This is done as follows:

(a) for S the spectrum of an Artinian k-algebra, (2.12) factors through $\tau^{\leq 0}\pi_*\iota^!\mathbf{K}_2$, so we obtain a *canonical* isomorphism $\mathcal{L}_{\tilde{e}}\big|_S \xrightarrow{\sim} \mathcal{L}_{\tilde{e}'}\big|_S$;

(b) there exists an isomorphism $\mathcal{L}_{\tilde{e}} \xrightarrow{\sim} \mathcal{L}_{\tilde{e}'}$ which restricts to the one in (a) for any S the spectrum of an Artinian k-algebra (§5.3.4-6 of *loc.cit.*).

We now claim that $\mathcal{L}_{\tilde{e}}\big|_{\mathcal{L}T}$ acquires a canonical multiplicative structure. Indeed, $\tilde{e}$ induces a morphism $X \times T \to \mathbf{K}_{\geq 2}[-1]$ of group sheaves. Given S-points t, t' of $\mathcal{L}T$ over the same point $x^I \in \mathrm{Ran}(X)$, we may view them both as maps $\overset{\circ}{D}_{x^I} \to X \times T$. There is a canonical homotopy between $\tilde{e}(t) + \tilde{e}(t')$ and $\tilde{e}(tt')$ as maps $\overset{\circ}{D}_{x^I} \to \mathbf{K}_{\geq 2}[-1]$. Under the map $\mathbf{K}_{\geq 2}\big|_{\overset{\circ}{D}_{x^I}}[-1] \to \iota^!\mathbf{K}_{\geq 2}$ of sheaves over D_{x^I}, we obtain a canonical homotopy between the corresponding sections of $\iota^!\mathbf{K}_{\geq 2}$; it gives rise to the desired multiplicative structure $\mathcal{L}_{\tilde{e}}\big|_t \otimes \mathcal{L}_{\tilde{e}}\big|_{t'} \xrightarrow{\sim} \mathcal{L}_{\tilde{e}}\big|_{tt'}$ under (2.12).

It remains to check that for two lifts $\tilde{e}$ and $\tilde{e}'$, the canonical isomorphism $\mathcal{L}_{\tilde{e}} \xrightarrow{\sim} \mathcal{L}_{\tilde{e}'}$ is compatible with the multiplicative structures on both sides. This amounts to checking that the following diagram of line bundles over $\mathcal{L}T \underset{\mathrm{Ran}(X)}{\times} \mathcal{L}T$ commutes:

$$\begin{tikzcd} \mathrm{mult}^*\,\mathcal{L}_{\tilde{e}} \arrow[r] \arrow[d] & \mathcal{L}_{\tilde{e}} \boxtimes \mathcal{L}_{\tilde{e}} \arrow[d] \\ \mathrm{mult}^*\,\mathcal{L}_{\tilde{e}'} \arrow[r] & \mathcal{L}_{\tilde{e}'} \boxtimes \mathcal{L}_{\tilde{e}'}. \end{tikzcd}$$

It suffices to test the commutativity over S the spectrum of an Artinian k-algebra. Note again that for such S, (2.12) factors through $\tau^{\leq 0}\pi_*\iota^!\mathbf{K}_2$, so the construction of the multiplicative structure does *not* require a lift of e. Therefore, we have equipped $\mathcal{L}$ with a canonical multiplicative structure over $\mathcal{L}T$.

Part (b) of the lemma is immediate from the above construction, applied to $S = \mathrm{Spec}(k)$ (or $\mathrm{Spec}(k')$ for a field extension $k \subset k'$). □

2.3. **Compatibility: torus case.**

[6] i.e., the group of k-points of $\mathcal{L}_x T$.

2.3.1. Fix a torus T. Recall the equivalence of Proposition 1.4:

$$\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_T) \xrightarrow{\sim} \theta(\Lambda_T). \tag{2.13}$$

The goal of this subsection is to prove:

Lemma 2.3. *The following diagram of Picard groupoids commutes functorially in T:*

$$\begin{tikzcd} \mathbf{CExt}(T, \mathbf{K}_2) \arrow[rr, "\Phi_T"] \arrow[dr, "(2.3)"'] & & \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_T) \arrow[dl, "(2.13)"] \\ & \theta(\Lambda_T) & \end{tikzcd} \tag{2.14}$$

Remark 2.4. Although Lemma 2.3 appears as the special case of Proposition 2.9 for $G = T$, its proof contains most of the technical difficulties.

2.3.2. *Notations.* Fix an object E of $\mathbf{CExt}(T, \mathbf{K}_2)$. We denote its image in $\theta^+(\Lambda_T)$ under (2.2) by $(q, \mathcal{L}_+^{(\lambda)}, c^+_{\mu,\nu})$, and its image under Φ_T by $\mathcal{L}$. The image of $\mathcal{L}$ in $\theta(\Lambda_T)$ will be denoted by $(q', \mathcal{L}^{(\lambda)}, c_{\mu,\nu})$. We ought to show:

(a) $q = q'$;
(b) there is a canonical system of isomorphisms:

$$\mathcal{L}_+^{(\lambda)} \xrightarrow{\sim} \mathcal{L}^{(\lambda)} \otimes \omega_X^{q(\lambda)} \tag{2.15}$$

which respects $c^+_{\mu,\nu}$ and $c_{\mu,\nu}$.

2.3.3. *Quadratic forms.* We first show $q = q'$ by checking that their bilinear forms κ and κ' agree. Fixing a closed point $x \in X$ and any co-character $\mu \in \Lambda_T$, we will show that $\kappa(-, \mu)$ and $\kappa'(-, \mu)$ define the same character $T(k') \to \mathbb{G}_m(k')$ for every field extension $k \subset k'$; this will imply that $\kappa = \kappa'$.[7]

We now further fix a uniformizer of the completed local ring $t \in \mathcal{O}_x$. This provides an isomorphism $k[\![t]\!] \xrightarrow{\sim} \mathcal{O}_x$, so we regard t^μ as an element of $T(\mathcal{K}_x)$. Consider the central extension (2.11) corresponding to $x \in X$. Pushing-out along the residue map $\mathbf{K}_2(\mathcal{K}_x) \to k^\times$, we obtain central extension:

$$0 \to k^\times \to E' \to T(\mathcal{K}_x) \to 0.$$

So the conjugation action of $T(\mathcal{O}_x)$ on the fiber of $E(\mathcal{K}_x) \to T(\mathcal{K}_x)$ at t^μ induces a map:

$$T(\mathcal{O}_x) \to k^\times. \tag{2.16}$$

We will calculate this map (and its variant for a field extension $k \subset k'$) in two ways.

Step 1. We first show that the map (2.16) is given by the composition:

$$T(\mathcal{O}_x) \xrightarrow{\mathrm{ev}} T(k) \xrightarrow{\kappa(-,\mu)} k^\times.$$

Indeed, recall from §2.1.3(a) that the composition $\mathbb{G}_m \underset{\mathbb{Z}}{\otimes} \mathbb{G}_m \xrightarrow{\lambda \otimes \mu} T \underset{\mathbb{Z}}{\otimes} T \xrightarrow{\mathrm{comm}} \mathbf{K}_2$ is the $\kappa(\lambda, \mu)$-multiple of the universal symbol. Thus the map:

$$\mathbb{G}_m(\mathcal{K}_x) \underset{\mathbb{Z}}{\otimes} \mathbb{G}_m(\mathcal{K}_x) \xrightarrow{\lambda \otimes \mu} T(\mathcal{K}_x) \underset{\mathbb{Z}}{\otimes} T(\mathcal{K}_x) \xrightarrow{\mathrm{comm}} \mathbf{K}_2(\mathcal{K}_x) \xrightarrow{\mathrm{res}} k^\times$$

[7] Indeed, for every $\lambda \in \Lambda_T$, suppose $z \rightsquigarrow z^{\kappa(\lambda,\mu)}$ and $z \rightsquigarrow z^{\kappa'(\lambda,\mu)}$ define the same map $\mathbb{G}_m(k') \to \mathbb{G}_m(k')$ for all field extension $k \subset k'$. By suitably choosing k', we can ensure that $(k')^\times$ contains an element of infinite order. Thus $\kappa(\lambda, \mu)$ agrees with $\kappa'(\lambda, \mu)$.

is the $\kappa(\lambda,\mu)$-multiple of the Contou-Carrère symbol $\{f,g\} := (f^{\mathrm{ord}(g)}/g^{\mathrm{ord}(f)})(0)$. Hence the conjugation action of $f \in \mathbb{G}_m(\mathcal{O}_x)$ (through λ) on E' is given by $e' \rightsquigarrow \{f,t\}^{\kappa(\lambda,\mu)}e'$. Note that $\{f,t\} = f(0)$, as required.

For a field extension $k \subset k'$, the above computation holds without modification.

Step 2. We now calculate the map (2.16) alternatively as follows. Recall the canonical multiplicative structure on $\mathcal{L}\big|_{\mathcal{L}T}$ from Lemma 2.1. It induces a *strong* $\mathcal{L}^+T$-equivariance structure on $\mathcal{L}$ (over Gr_T, c.f. [GL16, §7.3.4]) with respect to the trivial left $\mathcal{L}^+T$-action; in other words, the twisted product $\mathcal{L}\widetilde{\boxtimes}\mathcal{L}$ on the convolution Grassmannian $\widetilde{\mathrm{Gr}}_{T,X^2}$ is identified with the pullback of $\mathcal{L}^{(2)}$ along the action map $\widetilde{\mathrm{Gr}}_{T,X^2} \to \mathrm{Gr}_{T,X^2}$, in a way that is compatible with the factorization structure of $\mathcal{L}$.

Furthermore, its value at $\mathrm{Gr}^{\mu}_{T,x}$ is given by the conjugation action (2.16). We claim now that the map (2.16) is given by

$$T(\mathcal{O}_x) \xrightarrow{\mathrm{ev}} T(k) \xrightarrow{\kappa'(-,\mu)} k^{\times}.$$

Indeed, this follows from the fact that for a factorization line bundle $\mathcal{L}$ on Gr_T with associated bilinear form κ', every strong $\mathcal{L}^+T$-equivariance structure acts on $t^{\mu} \in \mathrm{Gr}_{T,x}$ through the composition $\mathcal{L}^+T \xrightarrow{\mathrm{ev}} T \xrightarrow{\kappa'(-,\mu)} \mathbb{G}_m$ (c.f. [GL16, §7.4]).

Again for a field extension $k \subset k'$, the above computation holds without modification. This finishes the proof that $\kappa = \kappa'$.

2.3.4. *Isomorphisms of line bundles.* We now construct the isomorphisms (2.15). The strategy is to first identify $\mathcal{L}^{(\lambda)}$ with the twist of $\mathcal{L}^{(\lambda)}_{+}$ by some power of the tangent sheaf $\mathcal{T}_X$, and then determine this power.

Step 1. Consider the diagonal embedding $\Delta : X \hookrightarrow X \times X$. Define $\mathcal{G}^{(\lambda)}$ as the $\mathbf{K}_2$-gerbe on $X \times X$ classifying a $\mathrm{pr}_2^* E$-torsor $\mathcal{P}_E$, together with an isomorphism $(\mathcal{P}_E)_T \xrightarrow{\sim} \mathcal{O}(\lambda\Delta)$. Then $\mathcal{G}^{(\lambda)}$ comes equipped with a neutralization γ over $X \times X - \Delta$. The line bundle $\mathcal{L}^{(\lambda)}$ arises from $(\mathcal{G}^{(\lambda)}, \gamma)$ by taking the residue along pr_1 (c.f. §2.2.2).

Let $X \times \mathbb{A}^1 \hookrightarrow \mathfrak{X} \to \mathbb{A}^1$ be the deformation of the diagonal embedding to the normal cone, constructed as the blow-up of $X \times X \times \mathbb{A}^1$ along the diagonally embedded subscheme $X \times \{0\}$, where we then remove the strict transform of $X \times X \times \{0\}$. It has the following features:

(a) $X \times \{t\} \hookrightarrow \mathfrak{X}\big|_t$ identifies with $X \hookrightarrow X \times X$ for $t \neq 0$;
(b) $X \times \{0\} \hookrightarrow \mathfrak{X}\big|_0$ identifies with the embedding of X as the zero section inside the total space of the tangent sheaf T_X.
(c) there is a canonical map $\mathfrak{X} \xrightarrow{\mathrm{pr}_1,\mathrm{pr}_2} X \times X$ which is identity for $t \neq 0$, and the canonical projection $T_X \xrightarrow{p,p} X \times X$ at $t = 0$.

Consider $\mathfrak{Z} := X \times \mathbb{A}^1$ as a divisor inside $\mathfrak{X}$. We define $\widetilde{\mathcal{G}}^{(\lambda)}$ as the $\mathbf{K}_2$-gerbe classifying a $\mathrm{pr}_2^* E$-torsor $\widetilde{\mathcal{P}}_E$ over $\mathfrak{X}$, together with an isomorphism $(\widetilde{\mathcal{P}}_E)_T \xrightarrow{\sim} \mathcal{O}(\lambda\mathfrak{Z})$. Note that $\widetilde{\mathcal{G}}^{(\lambda)}$ is equipped with a neutralization over $\mathfrak{X} - \mathfrak{Z}$, so we may take the residue along pr_1 to obtain a line bundle $\widetilde{\mathcal{L}}^{(\lambda)}$ over $X \times \mathbb{A}^1$.

Tautologically, $\widetilde{\mathcal{L}}^{(\lambda)}\big|_{X\times\{t\}}$ identifies with $\mathcal{L}^{(\lambda)}$ for $t \neq 0$. On the other hand, every line bundle on $X \times \mathbb{A}^1$ canonically identifies with the pullback of a line bundle from X. Thus, we obtain an isomorphism $\widetilde{\mathcal{L}}^{(\lambda)}\big|_{X\times\{t\}} \xrightarrow{\sim} \widetilde{\mathcal{L}}^{(\lambda)}\big|_{X\times\{0\}}$. This shows that $\mathcal{L}^{(\lambda)}$ arises from the residue of $(\mathcal{G}^{(\lambda)}_{T_X}, \gamma_{T_X})$ along $p : T_X \to X$, where:

(a) $\mathcal{G}^{(\lambda)}_{T_X}$ is the $\mathbf{K}_2$-gerbe on T_X classifying a p^*E-torsor $\mathcal{P}_E$, together with an isomorphism $(\mathcal{P}_E)_T \xrightarrow{\sim} \mathcal{O}(\lambda\{0\})$, where $\{0\}$ denotes the zero section $X \hookrightarrow T_X$; and

(b) γ_{T_X} is the tautological neutralization of $\mathcal{G}^{(\lambda)}_{T_X}$ over $T_X - \{0\}$.

Step 2. In the above description, suppose we replaced $p : T_X \to X$ by the trivial line bundle $\mathbb{A}^1_X \to X$; then the line bundle arising from taking the residue of the analogously defined pair $(\mathcal{G}^{(\lambda)}_{\mathbb{A}^1_X}, \gamma_{\mathbb{A}^1_X})$ would identify with $\mathcal{L}^{(\lambda)}_+$. Indeed, this follows from comparing the construction of §2.2.2 with that of §2.1.3(b).

We now explain an alternative way to arrive at $\mathcal{L}^{(\lambda)}$ via twisting the line bundle $\mathbb{A}^1_X \to X$ in the above construction. Consider the $\mathbb{G}_m$-action on $\mathbb{A}^1_X$ by scaling. The pair $(\mathcal{G}^{(\lambda)}_{\mathbb{A}^1_X}, \gamma_{\mathbb{A}^1_X})$ admits a $\mathbb{G}_m$-equivariance structure. Hence $L^{(\lambda)}_+$ (the total space of $\mathcal{L}^{(\lambda)}_+$) is equipped with a fiberwise $\mathbb{G}_m$-action. Since $\mathcal{G}^{(\lambda)}_{T_X}$ identifies with the twisted product $\mathcal{G}^0 \widetilde{\boxtimes} \mathcal{G}^{(\lambda)}_{\mathbb{A}^1_X}$ on the total space $T^{\times}_X \overset{\mathbb{G}_m}{\times} \mathbb{A}^1_X$ (where $\mathcal{G}^0$ denotes the trivial gerbe), we find $L^{(\lambda)} \xrightarrow{\sim} T^{\times}_X \overset{\mathbb{G}_m}{\times} L^{(\lambda)}_+$. In other words, suppose the fiberwise $\mathbb{G}_m$-action on $L^{(\lambda)}_+$ is given by some character $q_1(\lambda) \in \mathbb{Z}$, then there is a canonical isomorphism:

$$\mathcal{L}^{(\lambda)} \xrightarrow{\sim} \mathcal{T}_X^{q_1(\lambda)} \otimes \mathcal{L}^{(\lambda)}_+. \tag{2.17}$$

Step 3. We now calculate the character $q_1(\lambda)$.[8] It suffices to do so at a closed point $x \in X$. The line $L^{(\lambda)}_+\big|_{x\in X}$ admits a simple description as follows (c.f. §2.1.3). Evaluating E at $\mathbb{G}_{m,x} := \operatorname{Spec}(k[t, t^{-1}])$, we obtain an exact sequence:

$$0 \to \mathbf{K}_2(k[t, t^{-1}]) \to E(k[t, t^{-1}]) \to T(k_x[t, t^{-1}]) \to 0, \tag{2.18}$$

and consequently a $\mathbf{K}_2(k[t, t^{-1}])$-torsor $E(z)$ at every point $z \in T(k[t, t^{-1}])$. The line $L^{(\lambda)}_+\big|_{x\in X}$ is the $k^{\times}$-torsor induced from $E(t^{\lambda})$ along the residue map $\mathbf{K}_2(k[t, t^{-1}]) \to k^{\times}$.

To unburden the notation, we again use $L^{(\lambda)}_+$ to denote this line; the $\mathbb{G}_m(k)$-action on it also admits a simple description. Take $a \in \mathbb{G}_m(k)$, the action by $a^{q_1(\lambda)}$:

$$\cdot\, a^{q_1(\lambda)} : L^{(\lambda)}_+\big|_{x\in X} \xrightarrow{\sim} L^{(\lambda)}_+\big|_{x\in X} \tag{2.19}$$

is given as follows.

(a) Consider the scaling map $k[t, t^{-1}] \to k[t, t^{-1}]$, $t \rightsquigarrow t \cdot a$. It induces a group automorphism $E(k[t, t^{-1}]) \xrightarrow{a_*} E(k[t, t^{-1}])$, covering the analogously defined automorphism on $T(k[t, t^{-1}])$. In particular, we obtain a map $a_* : E(t^{\lambda}) \to E(t^{\lambda}a^{\lambda})$ (*incompatible* with the $\mathbf{K}_2(k[t, t^{-1}])$-torsor structures.)

After inducing to $k^{\times}$-torsors, we obtain a map *compatible* with the $k^{\times}$-torsor structures:

$$a_* : L^{(\lambda)}_+ \to L_+(t^{\lambda}a^{\lambda}) := E(t^{\lambda}a^{\lambda})_{k^{\times}},$$

since $a_* : \mathbf{K}_2(k[t, t^{-1}]) \to \mathbf{K}_2(k[t, t^{-1}])$ induces the identity on $k^{\times}$.

(b) On the other hand, every element in $T(k[t])$ admits a lift to $E(k[t])$, up to an element from $\mathbf{K}_2(k[t])$ (as follows from $\mathrm{R}^1 p_* \mathbf{K}_2 = 0$ for $p : \mathbb{A}^1_S \to S$, c.f. [BD01, §3.1]) Hence we have another map $E(t^{\lambda}) \to E(t^{\lambda}a^{\lambda})$, defined as right-multiplying by *any* lift of $a^{\lambda} \in T(k[t])$.

Inducing along $\mathbf{K}_2(k[t, t^{-1}]) \to k^{\times}$, we again obtain a map of $k^{\times}$-torsors:

$$R_{a^{\lambda}} : L^{(\lambda)}_+ \to L_+(t^{\lambda}a^{\lambda}).$$

Note that this map is independent of the choice of the lift.

(c) The automorphism (2.19) identifies with the composition $R^{-1}_{a^{\lambda}} \circ a_*$.

[8] Caution: we do not yet know that $q_1(\lambda)$ depends quadratically on λ.

Step 4. We shall now deduce two identities:

$$q_1(2\lambda) - \kappa(\lambda,\lambda) = 2\cdot q_1(\lambda) \tag{2.20}$$
$$4\cdot q_1(\lambda) = q_1(2\lambda) \tag{2.21}$$

The combination of these identities will show that $q_1(\lambda) = \frac{1}{2}\kappa(\lambda,\lambda) = q(\lambda)$. Then the desired isomorphism follows from (2.17).

Proof of (2.20). This follows from the mutiplicative structure on $E(k[t,t^{-1}])$. Indeed, consider the following commutative diagrams:

$$\begin{tikzcd}
L_+^{(2\lambda)} \arrow[r, "a_*"] \arrow[d, "\cong"] & L_+(t^{2\lambda}a^{2\lambda}) \arrow[d, "\cong"] \\
L_+^{(\lambda)}\otimes L_+^{(\lambda)} \arrow[r, "a_*\otimes a_*"] & L_+(t^{\lambda}a^{\lambda})\otimes L_+(t^{\lambda}a^{\lambda})
\end{tikzcd}
\qquad
\begin{tikzcd}
L_+^{(2\lambda)} \arrow[r, "a^{\kappa(\lambda,\lambda)}\cdot R_{a^{2\lambda}}"] \arrow[d, "\cong"] & L_+(t^{2\lambda}a^{2\lambda}) \arrow[d, "\cong"] \\
L_+^{(\lambda)}\otimes L_+^{(\lambda)} \arrow[r, "R_{a^\lambda}\otimes R_{a^\lambda}"] & L_+(t^{\lambda}a^{\lambda})\otimes L_+(t^{\lambda}a^{\lambda})
\end{tikzcd}$$

where vertical arrows witness the multiplicativity of $\mathcal{L}_+^{(\lambda)}$. The first diagram commutes because a_* defines a group homomorphism on $E(k[t,t^{-1}])$. The second diagram commutes (note the factor $a^{\kappa(\lambda,\lambda)}$) because it calculates the commutator $\mathrm{comm}(a^\lambda, t^\lambda)\in\mathbf{K}_2(k[t,t^{-1}])$, whose residue identifies with $a^{\kappa(\lambda,\lambda)}$.

Now, tracing through the horizontal arrows gives rise to the identity $a^{q_1(2\lambda)-\kappa(\lambda,\lambda)} = a^{2\cdot q_1(\lambda)}$ in $k^\times$. Since the same calculation is valid for any field extension $k\subset k'$, we obtain (2.20). □

Proof of (2.21). This follows from the functoriality of $E(k[t,t^{-1}])$ with respect to the double covering map $\mathrm{sq}(t) = t^2$ on $k[t,t^{-1}]$. Note that $\mathrm{sq}_* : E(k[t,t^{-1}])\to E(k[t,t^{-1}])$ induces a *quadratic* map of $k^\times$-torsors[9]:

$$\mathrm{sq}_* : L_+^{(\lambda)} \to L_+^{(2\lambda)}.$$

On the other hand, we have the following commutative diagrams:

$$\begin{tikzcd}
L_+^{(\lambda)} \arrow[r, "(a^2)_*"] \arrow[d, "\mathrm{sq}_*"] & L_+(t^{\lambda}a^{2\lambda}) \arrow[d, "\mathrm{sq}_*"] \\
L_+^{(2\lambda)} \arrow[r, "a_*"] & L_+(t^{2\lambda}a^{2\lambda})
\end{tikzcd}
\qquad
\begin{tikzcd}
L_+^{(\lambda)} \arrow[r, "R_{a^{2\lambda}}"] \arrow[d, "\mathrm{sq}_*"] & L_+(t^{\lambda}a^{2\lambda}) \arrow[d, "\mathrm{sq}_*"] \\
L_+^{(2\lambda)} \arrow[r, "R_{a^{2\lambda}}"] & L_+(t^{2\lambda}a^{2\lambda})
\end{tikzcd}$$

The first diagram commutes tautologically. The second diagram commutes because $a^{2\lambda}$ belongs to the subgroup $T(k)\hookrightarrow T(k[t,t^{-1}])$, and we may first lift $a^{2\lambda}$ to $E(k)$ so that its image in $E(k[t,t^{-1}])$ is fixed by the automorphism sq_*. Tracing through the horizontal maps and using the quadraticity of vertical maps, we find $a^{4\cdot q_1(\lambda)} = a^{q_1(2\lambda)}$ in $k^\times$. Again because the same calculation is valid for any field extension $k\subset k'$, we obtain (2.21). □

□(Lemma 2.3)

2.4. **Compatibility: general case.**

[9] i.e., the $k^\times$-action on the two lines intertwines $k^\times\to k^\times$, $a\rightsquigarrow a^2$.

2.4.1. We now return to the general case of a reductive group G. Appealing to the equivalence (2.13), we obtain a functor:

$$\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_G) \xrightarrow{\mathrm{res}} \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_T) \xrightarrow{\sim} \theta(\Lambda_T) \to Q(\Lambda_T, \mathbb{Z}). \tag{2.22}$$

Proposition 2.5. *Suppose G is semisimple and simply connected. Then (2.22) defines an equivalence:* $\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_G) \xrightarrow{\sim} Q(\Lambda_T, \mathbb{Z})^W$.

In this subsection, we will first prove Proposition 2.5, and then use it to deduce the general compatibility result between Gaitsgory functor Φ_G and the Brylinski–Deligne classification.

2.4.2. We use the notation $\mathbf{Pic}^e(\mathrm{Gr}_G)$ to denote the Picard groupoid of line bundles on Gr_G together with a rigidification at the unit section $e : \mathrm{Ran}(X) \hookrightarrow \mathrm{Gr}_G$; the notation $\mathbf{Pic}^e(\mathrm{Gr}_{G,X^I})$ carries an analogous meaning. Since factorization line bundles on $\mathrm{Ran}(X)$ are canonically trivial (c.f. Lemma 1.3), we have a forgetful functor $\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_G) \to \mathbf{Pic}^e(\mathrm{Gr}_G)$.

2.4.3. We first prove Proposition 2.5 in the case where G is *simple* and simply connected. We note that in this case, the abelian group $Q(\Lambda_T, \mathbb{Z})^W$ is isomorphic to $\mathbb{Z}$, where a generator is given by the *minimal* W-invariant quadratic form $q_{\min}$, uniquely specified by the property that $q(\alpha) = 1$ for any short coroot α.

We fix a point $x \in X$. The calculation of Picard schemes $\mathbf{Pic}^e(\mathrm{Gr}_{G,X^I})$ in [Zh16, §3.4] proves that there are isomorphisms:

$$\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_G) \xrightarrow{\sim} \mathbf{Pic}^e(\mathrm{Gr}_G) \xrightarrow{\sim} \mathbf{Pic}^e(\mathrm{Gr}_{G,x}), \tag{2.23}$$

given by pulling back along $\mathrm{Gr}_{G,x} \hookrightarrow \mathrm{Gr}_G$. On the other hand, the result of G. Faltings [Fa03] shows that $\mathbf{Pic}^e(\mathrm{Gr}_{G,x})$ is also isomorphic to $\mathbb{Z}$ (in particular, it is discrete), and the generator of $\mathbf{Pic}^e(\mathrm{Gr}_{G,x})$ is a certain line bundle $\mathcal{L}_{\min}$ satisfying the following property:

(*) Let $\mathcal{L}_{\det}$ be the determinant line bundle on $\mathrm{Gr}_{G,x}$, whose fiber at an S-point $(\mathcal{P}_G, \mathcal{P}_G\big|_{\overset{\circ}{D}_x} \xrightarrow{\sim} \mathcal{P}_G^0)$ is the relative determinant of the lattices $\mathfrak{g}_{\mathcal{P}_G}, \mathfrak{g}_{\mathcal{P}_G^0} \subset \mathfrak{g}(\mathcal{K}_x)$. Then there is an isomorphism $(\mathcal{L}_{\min})^{\otimes 2\check{h}} \xrightarrow{\sim} \mathcal{L}_{\det}$.

In order to show that (2.22) is an isomorphism onto $Q(\Lambda_T, \mathbb{Z})^W$, it suffices to show that for some nonzero integer d, the image of $(\mathcal{L}_{\min})^{\otimes d}$ (regarded as an element in $\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_G)$ via (2.23)) equals $d \cdot q$. We will prove this statement for $d = 2\check{h}$ by calculating the image of $\mathcal{L}_{\det}$.

Note that $\mathcal{L}_{\det}$ has a natural factorization structure (c.f. [GL16, §5.2.1]). By tracing through the functors in (2.22), we see that its image is the quadratic form $q_{\det}$ whose associated bilinear form $\kappa_{\det}$ equals:

$$\kappa_{\det}(\lambda, \mu) = \sum_{\check{\alpha} \in \Phi} \langle \lambda, \check{\alpha} \rangle \langle \mu, \check{\alpha} \rangle = \mathrm{Kil}(\lambda, \mu),$$

where Kil stands for the Killing form. On the other hand, $\check{h}$ is defined so that $\mathrm{Kil} = 2\check{h} \cdot \kappa_{\min}$. Thus $q_{\det} = 2\check{h} \cdot q_{\min}$ as desired.

2.4.4. In order to handle the general case, we first note a cohomological vanishing result that will also be useful later. We continue to fix a k-point $x \in X$. Recall that for a dominant cocharacter $\lambda \in \Lambda_G^+$, we have the affine Schubert variety $\mathrm{Gr}_{G,x}^{\leq \lambda} \hookrightarrow \mathrm{Gr}_{G,x}$ such that $\mathrm{Gr}_{G,x}$ is isomorphic to the infinite union $\operatorname*{colim}_{\lambda \in \Lambda_T^+} \mathrm{Gr}_{G,x}^{\leq \lambda}$. When G is semisimple and simply connected, each $\mathrm{Gr}_{G,x}^{\leq \lambda}$ is integral.

Lemma 2.6. *Suppose G is semisimple and simply connected. Then for any $\lambda \in \Lambda_G^+$, we have* $\mathrm{H}^i(\mathrm{Gr}_{G,x}^{\leq \lambda}, \mathcal{O}) = 0$ *for* $i \geq 1$.

Proof. Let I denote the Iwahori subgroup of $\mathcal{L}_x^+ G$ and $\mathrm{Fl}_{G,x} := \mathcal{L}_x G/I$ be the affine flag variety. The I-orbits of $\mathrm{Fl}_{G,x}$ are parametrized by the affine Weyl group W^{aff}. Let $\mathrm{Fl}^w_{G,x}$ denote the orbit corresponding to $w \in W^{\mathrm{aff}}$ and $\mathrm{Fl}^{\leq w}_{G,x}$ its closure. We note that the projection $\mathrm{Fl}_{G,x} \to \mathrm{Gr}_{G,x}$ is a flat-locally trivial fiber bundle with typical fiber G/B. Furthermore, for any $\lambda \in \Lambda_G^+$, there is a Cartesian square:

$$\begin{tikzcd} \mathrm{Fl}^{\leq w}_{G,x} \arrow[r, hook] \arrow[d] & \mathrm{Fl}_{G,x} \arrow[d] \\ \mathrm{Gr}^{\leq \lambda}_{G,x} \arrow[r, hook] & \mathrm{Gr}_{G,x} \end{tikzcd}$$

where w is the longest element in the double coset of λ, after we identify Λ_G^+ with $W\backslash W^{\mathrm{aff}}/W$. Since $k \xrightarrow{\sim} \mathrm{R}\,\Gamma(G/B, \mathcal{O})$, we reduce the proof to showing $k \xrightarrow{\sim} \mathrm{R}\,\Gamma(\mathrm{Fl}^{\leq w}_{G,x}, \mathcal{O})$.

We now make an argument similar to that for finite dimensional Schubert varieties. Namely, for each simple (affine) reflection $s \in W^{\mathrm{aff}}$, we let $P_s := I \cup (IsI)$ denote the corresponding minimal parahoric subgroup. Suppose $w = s_1 \cdots s_l$ is an reduced expression. Then we have an *affine* Bott–Samelson resolution:

$$\widetilde{\mathrm{Fl}}{}^{\leq w}_G := P_{s_1} \overset{I}{\times} P_{s_2} \overset{I}{\times} \cdots \overset{I}{\times} P_{s_l}/I \to \mathrm{Fl}^{\leq w}_G, \tag{2.24}$$

where the I-superscripts indicate quotients by anti-diagonal actions. Since each P_s/I is isomorphic to $\mathbb{P}^1$, the scheme $\widetilde{\mathrm{Fl}}{}^{\leq w}_G$ is an iterated $\mathbb{P}^1$-bundle. Thus, we reduce to showing that $\mathcal{O}_{\widetilde{\mathrm{Fl}}{}^{\leq w}_G}$ has vanishing higher direct image along (2.24), and this follows from the same proof as the usual Bott–Samelson resolution, c.f. [Br04, Theorem 2.2.3]. ☐

Remark 2.7. Lemma 2.6 can be seen as an affine version of the Borel–Weil–Bott theorem and is likely to be known, but the authors could not find a reference.

2.4.5. We now prove Proposition 2.5 in the general case. Suppose G has simple factors $\{G_j\}_{j\in J}$. It suffices to prove that pulling back along the factors $\mathrm{Gr}_{G_j} \hookrightarrow \mathrm{Gr}_G$ defines an equivalence of Picard groupoids:

$$\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_G) \xrightarrow{\sim} \prod_{j\in J} \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G_j}). \tag{2.25}$$

Note that this morphism fits into a commutative diagram of Picard groupoids:

$$\begin{tikzcd} \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_G) \arrow[r] \arrow[d, "(2.25)"] & \mathbf{Pic}^{e}(\mathrm{Gr}_G) \arrow[r, "(b)"] \arrow[d, "(c)"] & \mathbf{Pic}^{e}(\mathrm{Gr}_{G,x}) \arrow[d, "(a)"] \\ \prod_{j\in J} \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G_j}) \arrow[r, "\sim"] & \prod_{j\in J} \mathbf{Pic}^{e}(\mathrm{Gr}_{G_j}) \arrow[r, "\sim"] & \prod_{j\in J} \mathbf{Pic}^{e}(\mathrm{Gr}_{G_j,x}) \end{tikzcd}$$

where the lower row consists of equivalences, c.f. (2.23). We note that the cohomological vanishing Lemma 2.6 for $i = 1$ implies that (a) is an equivalence.[10] That (b) is an equivalence follows from [Zh16, Lemma 3.4.2] and the proof of [Zh16, Lemma 3.4.3]. Together, these facts imply that (c) is an equivalence.

[10]Recall: suppose $X, Y \in \mathbf{Sch}_{/k}$ are connected schemes of finite type with base points, and X is integral, projective with $\mathrm{H}^1(X, \mathcal{O}_X) = 0$. Then $\mathbf{Pic}^e(X) \times \mathbf{Pic}^e(Y) \xrightarrow{\sim} \mathbf{Pic}^e(X \times Y)$ (see [Ha13, Exercise III.12.6]).

2.4.6. Finally, we argue that the left square is Cartesian, which would imply that (2.25) is an equivalence. Concretely, this means that given a rigidified line bundle $\mathcal{L}$ over Gr_G (which passes to $\boxtimes_{j\in J}\mathcal{L}_j$ over $\prod_{j\in J}\mathrm{Gr}_{G_j}$ via the equivalence (c)), the datum needed to upgrade it to a factorization structure on $\mathcal{L}$:

$$\varphi : \mathcal{L}^{(2)}\big|_{X^2-\Delta} \xrightarrow{\sim} \mathcal{L}^{(1)} \boxtimes \mathcal{L}^{(1)}$$

is equivalent to that of factorization structures φ_j on each $\mathcal{L}_j$. We note that the collection $\{\varphi_j\}_{j\in J}$ defines a factorization structure $\boxtimes_{j\in J}\varphi_j$ on $\mathcal{L}$ and conversely a factorization structure φ on $\mathcal{L}$ defines φ_j by restriction to the jth unit section $X^2 \underset{X^2}{\times} \cdots \underset{X^2}{\times} \mathrm{Gr}_{G_j,X^2} \underset{X^2}{\times} \cdots \underset{X^2}{\times} X^2 \hookrightarrow \mathrm{Gr}_{G,X^2}$. Thus it remains to show:

Claim 2.8. Any $\mathcal{L} \in \mathbf{Pic}^e(\mathrm{Gr}_G)$ has at most one factorization structure compatible with its rigidification.

Indeed, any two such factorization structures differ by an automorphism β of $\mathcal{L}^{(2)}\big|_{X^2-\Delta}$ that restricts to identity along the unit section. Since $\mathrm{Gr}_{G,X^2}\big|_{X^2-\Delta}$ is an ind-integral ind-scheme over $X^2-\Delta$, it suffices to show that β becomes the identity after restricting to the fibers at k-points of $X^2-\Delta$. The latter follows from the discreteness of $\mathbf{Pic}^e(\mathrm{Gr}_{G,x}\times\mathrm{Gr}_{G,y})$, which in turn follows from that of $\mathbf{Pic}^e(\mathrm{Gr}_{G,x})$ and Lemma 2.6. $\square$(Proposition 2.5)

2.4.7. For a semisimple and simply connected group G, we obtain a map:

$$Q(\Lambda_T,\mathbb{Z})^W \to \theta(\Lambda_T)$$

by first lifting an element of $Q(\Lambda_T,\mathbb{Z})^W$ to $\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_G)$ using the isomorphism of Proposition 2.5, and then mapping to $\theta(\Lambda_T)$. By Lemma 2.3, the above functor identifies with (2.5).

2.4.8. Recall the Picard groupoid $\theta_G(\Lambda_T)$ of §2.1. We will define a functor:

$$\Psi_G : \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_G) \to \theta_G(\Lambda_T) \tag{2.26}$$

Given $\mathcal{L} \in \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_G)$, we will construct a theta datum $(q,\mathcal{L}^{(\lambda)},c_{\lambda,\mu})$ for Λ_T as well as an isomorphism φ of two corresponding theta data for $\Lambda_{\widetilde{T}_{\mathrm{der}}}$.

Indeed, $(q,\mathcal{L}^{(\lambda)},c_{\lambda,\mu})$ is the image of $\mathcal{L}$ under the first two maps of (2.22). On the other hand, $\mathcal{L}$ restricts to a factorization line bundle on $\mathrm{Gr}_{\widetilde{G}_{\mathrm{der}}}$; under the same two maps, we obtain a theta datum $(q\big|_{\Lambda_{\widetilde{T}_{\mathrm{der}}}},\widetilde{\mathcal{L}}^{(\lambda)},\widetilde{c}_{\lambda,\mu})$. By §2.4.1, this is the theta datum associated to $q\big|_{\Lambda_{\widetilde{T}_{\mathrm{der}}}}$ under (2.5). Therefore, we obtain φ from the commutativity datum of the diagram:

$$\begin{tikzcd}
\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_G) \arrow[r, "\mathrm{res}"] \arrow[d] & \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_T) \arrow[r, "\sim"] \arrow[d] & \theta(\Lambda_T) \arrow[d] \\
\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{\widetilde{G}_{\mathrm{der}}}) \arrow[r, "\mathrm{res}"] & \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{\widetilde{T}_{\mathrm{der}}}) \arrow[r, "\sim"] & \theta(\Lambda_{\widetilde{T}_{\mathrm{der}}}).
\end{tikzcd}$$

2.4.9. We now state the main compatibility result, generalizing Lemma 2.3:

Proposition 2.9. *The following diagram of Picard groupoids is canonically commutative:*

$$\begin{tikzcd}
\mathbf{CExt}(G,\mathbf{K}_2) \arrow[rr, "\Phi_G"] \arrow[dr, "\Phi_{\mathrm{BD}}"'] & & \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_G) \arrow[dl, "\Psi_G"] \\
& \theta_G(\Lambda_T) &
\end{tikzcd} \tag{2.27}$$

Proof. Given a central extension of G by $\mathbf{K}_2$, we have to construct an isomorphism between two elements of $\theta(\Lambda_T)$ and check that it respects the isomorphism denoted by φ. The isomorphism comes from the commutativity datum of Lemma 2.3, and the required compatibility follows from its functoriality with respect to the map of tori $\widetilde{T}_{\mathrm{der}} \to T$. □

Let us describe the functoriality of the commutativity datum in Proposition 2.9. Given a morphism $\alpha : (G', T') \to (G, T)$ between pairs of a reductive group together with a chosen maximal torus, there is a pullback functor

$$\alpha^* : \theta_G(\Lambda_T) \to \theta_{G'}(\Lambda_{T'}).$$

The morphisms Φ_{BD} and Ψ_G are canonically compatible with this pullback. For each $E \in \mathbf{CExt}(G, \mathbf{K}_2)$ whose pullback to $\mathbf{CExt}(G', \mathbf{K}_2)$ is denoted by E', one deduces from the functoriality in Lemma 2.3 that the following diagram commutes:

$$\begin{tikzcd} \Phi_{\mathrm{BD}}(E') \arrow[r, "\sim"] \arrow[d, "\cong"] & \Psi_{G'}\Phi_{G'}(E') \arrow[d, "\cong"] \\ \alpha^*\Phi_{\mathrm{BD}}(E) \arrow[r, "\sim"] & \alpha^*\Psi_G\Phi_G(E). \end{tikzcd}$$

Here, the horizontal arrows are the commutativity data exhibited in Proposition 2.9.

3. The main theorem

This section is devoted to the proof that Gaitsgory's functor Φ_G is an equivalence of categories. We assume $\mathrm{char}(k) \nmid N_G$ so that the functor Φ_G is well-defined.

3.1. **Statement and reduction.**

3.1.1. Let us first state the main theorem of the paper.

Theorem 3.1. *Suppose* $\mathrm{char}(k) \nmid N_G$. *Then the functor* Φ_G (2.7) *is an equivalence of Picard groupoids.*

Using the commutativity of (2.27) and the fact that Φ_{BD} is an equivalence, we have already obtained some special cases of Theorem 3.1:

(a) the case $G = T$ is a torus follows from Proposition 1.4, as $\theta_G(\Lambda_T)$ becomes $\theta(\Lambda_T)$;
(b) the case G semisimple, simply connected follows from Proposition 2.5, as $\theta_G(\Lambda_T)$ becomes the (discrete) abelian group $Q(\Lambda_T, \mathbb{Z})^W$.

3.1.2. We now perform a reduction of Theorem 3.1 to the case where G_{der} is simply connected. Choose an exact sequence of groups:

$$1 \to T_2 \to \widetilde{G} \to G \to 1, \tag{3.1}$$

where T_2 is a torus, and $\widetilde{G}$ is a reductive group whose derived subgroup is simply connected. The sequence (3.1) is called a *z-extension*, c.f. [MS82, Proposition 3.1]. Consider the simplicial system $\widetilde{G} \times T_2^{\bullet}$, where the nth simplex is given by $\widetilde{G} \times T_2^{\times n}$ and the boundary maps are multiplications. Since T_2 is central in $\widetilde{G}$, these multiplication maps define morphisms of algebraic groups. As a consequence, we obtain a simplicial system of prestacks $\mathrm{Gr}_{\widetilde{G} \times T_2^{\bullet}}$ over $\mathrm{Ran}(X)$. Appealing to [Ga18, Corollary 5.2.7], the Picard groupoid $\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_G)$ identifies with the limit of the co-simplicial system $\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{\widetilde{G} \times T_2^{\bullet}})$.

Remark 3.2. The cited result follows from h-descent of line bundles for *derived* schemes. The proof given there uses h-descent of ind-coherent sheaves, which has been established by Gaitsgory [Ga11, Theorem 8.2.2] in the context where $\mathrm{char}(k) = 0$ (see also [GR17, Chapter 4, Proposition 7.2.2] for a more detailed account).

However, invoking ind-coherent sheaves is unnecessary for this application: h-descent of line bundles is also a consequence of a theorem of Halpern-Leistner–Preygel [HLP14, Theorem 3.3.1], which is valid for derived schemes over any Noetherian base scheme.

Lemma 3.3. *The canonical map of Picard groupoids is an equivalence:*

$$\mathbf{CExt}(G, \mathbf{K}_2) \xrightarrow{\sim} \lim \mathbf{CExt}(\widetilde{G} \times T_2^{\bullet}, \mathbf{K}_2).$$

Proof. We argue that the Picard groupoid of (not necessarily central) extensions $\mathbf{Ext}(G, \mathbf{K}_2)$ maps isomorphically to $\lim \mathbf{Ext}(\widetilde{G} \times T_2^{\bullet}, \mathbf{K}_2)$; the result would follow since a $\mathbf{K}_2$-extension of G is central if and only if its pullback to each $\widetilde{G} \times T_2^{\bullet}$ is central.

Since $\mathbf{Ext}(G, \mathbf{K}_2)$ identifies with homomorphisms from G to $\mathrm{B}\,\mathbf{K}_2$, it suffices to show that G identifies with $\mathrm{colim}(\widetilde{G} \times T_2^{\bullet})$ in the category of Zariski sheaves of groups (in spaces). This in turn follows from:

(a) the forgetful functor from Zariski sheaves of groups to plain Zariski sheaves is conservative and commutes with geometric realizations;
(b) G identifies with $\mathrm{colim}(\widetilde{G} \times T_2^{\bullet})$ in the category of plain Zariski sheaves, since every T_2-torsor is Zariski-locally trivial (Hilbert 90). ☐

In other words, Theorem 3.1 for G follows from the same result for each $\widetilde{G} \times T_2^{\bullet}$. In proving Theorem 3.1, we may thus assume that G_{der} is simply connected.

3.2. Proof of Theorem 3.1 for G_{der} simply connected.

3.2.1. We now prove Theorem 3.1 in the case that G_{der} is simply connected. Let $T_1 := G/G_{\mathrm{der}}$. Then the fiber of $\theta_G(\Lambda_T) \to Q(\Lambda_{T_{\mathrm{der}}}, \mathbb{Z})^W$ identifies with $\theta(\Lambda_{T_1})$. Let $\mathbf{Pic}^{\mathrm{fact}}_{q_{\mathrm{der}}=0}(\mathrm{Gr}_G)$ be the full subgroupoid of $\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_G)$, consisting of objects whose images vanish under the following composition:

$$\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_G) \to \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G_{\mathrm{der}}}) \xrightarrow{(2.22)} Q(\Lambda_{T_{\mathrm{der}}}, \mathbb{Z}).$$

We then have a commutative diagram of Picard groupoids:

$$\begin{tikzcd}
& & \mathbf{CExt}(G; \mathbf{K}_2) \arrow[dl, bend right, "\Phi_G"'] \arrow[ddl, bend left, "\Phi_{\mathrm{BD}}", "\cong"'] \\
\mathbf{Pic}^{\mathrm{fact}}_{q_{\mathrm{der}}=0}(\mathrm{Gr}_G) \arrow[r, hook] \arrow[d] & \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_G) \arrow[d, "\Psi_G"'] & \\
\theta(\Lambda_{T_1}) \arrow[r, hook] & \theta_G(\Lambda_T) \arrow[r] & Q(\Lambda_{T_{\mathrm{der}}}, \mathbb{Z})^W.
\end{tikzcd}$$

Here, Ψ_G is the functor (2.26). Inspecting this diagram, we see that it suffices to show that the first vertical map:

$$\mathbf{Pic}^{\mathrm{fact}}_{q_{\mathrm{der}}=0}(\mathrm{Gr}_G) \to \theta(\Lambda_{T_1}) \tag{3.2}$$

is an equivalence.

3.2.2. Consider the projection $\mathfrak{p} : \mathrm{Gr}_G \to \mathrm{Gr}_{T_1}$. It defines a pullback functor

$$\mathfrak{p}^* : \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{T_1}) \to \mathbf{Pic}^{\mathrm{fact}}_{q_{\mathrm{der}}=0}(\mathrm{Gr}_G) \tag{3.3}$$

such that the composition:

$$\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{T_1}) \xrightarrow{\mathfrak{p}^*} \mathbf{Pic}^{\mathrm{fact}}_{q_{\mathrm{der}}=0}(\mathrm{Gr}_G) \xrightarrow{(3.2)} \theta(\Lambda_{T_1})$$

canonically identifies with the equivalence (2.13). It therefore suffices to show that (3.3) is an equivalence.

3.2.3. We note that (3.3) factors through the full subcategory

$$\mathbf{Pic}^{\mathrm{fact}}_{\natural}(\mathrm{Gr}_G) \hookrightarrow \mathbf{Pic}^{\mathrm{fact}}_{q_{\mathrm{der}}=0}(\mathrm{Gr}_G) \tag{3.4}$$

of factorization line bundles on Gr_G which are trivial along fibers of $\mathfrak{p}$ over k-points. In the rest of this subsection, we shall show that

(a) the containment (3.4) is an equivalence.
(b) pullback along $\mathfrak{p}$ defines an equivalence

$$\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{T_1}) \to \mathbf{Pic}^{\mathrm{fact}}_{\natural}(\mathrm{Gr}_G). \tag{3.5}$$

The combination of these two statements will imply Theorem 3.1.

3.2.4. In order to prove the above statements, we first study the geometric properties of the projection $\mathfrak{p}$.

Lemma 3.4. *The map $\mathfrak{p}$ realizes Gr_G as an étale locally trivial $\mathrm{Gr}_{G_{\mathrm{der}}}$-bundle over Gr_{T_1}.*

In other words, for every affine scheme $S \to \mathrm{Gr}_{T_1}$, there is an étale cover $\widetilde{S} \to S$ and an isomorphism $\mathrm{Gr}_G \underset{\mathrm{Gr}_{T_1}}{\times} \widetilde{S} \xrightarrow{\sim} \mathrm{Gr}_{G_{\mathrm{der}}} \underset{\mathrm{Ran}(X)}{\times} \widetilde{S}$.

Proof of Lemma 3.4. We first show that $G \to T_1$ splits. Indeed, the maximal (split) torus $T \subset G$ surjects onto T_1, so it suffices to show that the kernel $T \cap G_{\mathrm{der}}$ is connected. The latter follows since $T \cap G_{\mathrm{der}}$ is a maximal torus of G_{der}.

Given an S-point $S \xrightarrow{\gamma} \mathrm{Gr}_{T_1}$, we denote by $S \xrightarrow{\gamma_0} \mathrm{Gr}_{T_1}$ the "neutral point" corresponding to γ, i.e., the composition $S \xrightarrow{\gamma} \mathrm{Gr}_{T_1} \xrightarrow{\pi} \mathrm{Ran}(X) \hookrightarrow \mathrm{Gr}_{T_1}$. Since $\mathrm{Gr}_G \underset{\mathrm{Gr}_{T_1}, \gamma_0}{\times} S$ identifies with $\mathrm{Gr}_{\widetilde{G}_{\mathrm{der}}} \underset{\mathrm{Ran}(X)}{\times} S$, it suffices to produce an isomorphism:

$$\mathrm{Gr}_G \underset{\mathrm{Gr}_{T_1}, \gamma}{\times} \widetilde{S} \xrightarrow{\sim} \mathrm{Gr}_G \underset{\mathrm{Gr}_{T_1}, \gamma_0}{\times} \widetilde{S} \tag{3.6}$$

after passing to some étale cover $\widetilde{S} \to S$.

We choose $\widetilde{S} \to S$ such that the elements $\gamma, \gamma_0 \in \mathrm{Maps}_{/\mathrm{Ran}(X)}(\widetilde{S}, \mathrm{Gr}_{T_1})$ differ by the action of some $\alpha \in \mathrm{Maps}_{/\mathrm{Ran}(X)}(\widetilde{S}, \mathfrak{L}T_1)$ (this is possible, for example, by lifting $S \to \mathrm{Gr}_{T_1}$ to $\widetilde{S} \to \mathfrak{L}T_1$). The above discussion shows that we have a splitting of the canonical projection $\mathfrak{L}G \to \mathfrak{L}T_1$. Hence α can be lifted to an element $\widetilde{\alpha} \in \mathrm{Maps}_{/\mathrm{Ran}(X)}(\widetilde{S}, \mathfrak{L}G)$. The equivariance property of $\mathfrak{p}$ shows that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Gr}_G \underset{\mathrm{Ran}(X)}{\times} \widetilde{S} \arrow[r, "\mathrm{act}_{\widetilde{\alpha}}"] \arrow[d] & \mathrm{Gr}_G \underset{\mathrm{Ran}(X)}{\times} \widetilde{S} \arrow[d] \\
\mathrm{Gr}_{T_1} \underset{\mathrm{Ran}(X)}{\times} \widetilde{S} \arrow[r, "\mathrm{act}_{\alpha}"] & \mathrm{Gr}_{T_1} \underset{\mathrm{Ran}(X)}{\times} \widetilde{S}
\end{tikzcd}$$

Since act_α transforms the section $\gamma : \widetilde{S} \to \mathrm{Gr}_{T_1} \underset{\mathrm{Ran}(X)}{\times} \widetilde{S}$ to γ_0, we obtain the required isomorphism (3.6) as $\mathrm{act}_{\widetilde{\alpha}} \underset{\mathrm{act}_\alpha}{\times} \mathrm{id}_{\widetilde{S}}$. $\square$

3.2.5. *Proof of* (a). We now show that every $\mathcal{L} \in \mathbf{Pic}^{\mathrm{fact}}_{q_{\mathrm{der}}=0}(\mathrm{Gr}_G)$ is fiberwise trivial along the projection $\mathfrak{p} : \mathrm{Gr}_G \to \mathrm{Gr}_{T_1}$. Since the question concerns only points on Gr_{T_1}, it suffices to show that the base change of $\mathcal{L}$ to the subscheme $X^{(\lambda_1,\cdots,\lambda_{|I|})} \hookrightarrow \mathrm{Gr}_{T_1,X^I}$[11] is fiberwise trivial.

We write $\underline{\mathbf{P}}^{(\lambda^I)}$ for the étale sheaf of relative Picard group of $\mathrm{Gr}_{G,X^I} \underset{\mathrm{Gr}_{T_1,X^I}}{\times} X^{(\lambda_1,\cdots,\lambda_{|I|})}$ over $X^{(\lambda_1,\cdots,\lambda_{|I|})}$, i.e., it associates to every étale map $V \to X^{(\lambda_1,\cdots,\lambda_{|I|})}$ the abelian group $\mathbf{Pic}(\mathrm{Gr}_{G,X^n} \underset{\mathrm{Gr}_{T_1,X^n}}{\times} V)/\mathbf{Pic}(V)$. Thus $\mathcal{L}$ defines a global section $l^{(\lambda^I)}$ of $\underline{\mathbf{P}}^{(\lambda^I)}$ for every n-tuple λ^I. The goal is to show that all $l^{(\lambda^I)}$ vanish.

3.2.6. Recall the computation of the étale sheaf of relative Picard groups $\underline{\mathbf{Pic}}(\mathrm{Gr}_{G_{\mathrm{der}},X^I}/X^I)$ in [Zh16, §3.4]. It fits into an exact sequence of sheaves of abelian groups over X^I:

$$0 \to \underline{\mathbf{Pic}}(\mathrm{Gr}_{G_{\mathrm{der}},X^I}/X^I) \to \boxtimes_{i\in I}\underline{A}_X \to \bigoplus_{|J|=|I|-1} (\Delta_{I\twoheadrightarrow J})_* \boxtimes_{j\in J} \underline{A}_X.$$

Here, A denotes the abelian group $\mathbb{Z}^{\times\,\mathrm{rank}(G_{\mathrm{der}})}$, and $\underline{A}_X$ is its associated constant sheaf of groups over X. Lemma 3.4 shows that the sheaf $\underline{\mathbf{P}}^{(\lambda^I)}$ is étale locally isomorphic to $\underline{\mathbf{Pic}}(\mathrm{Gr}_{G_{\mathrm{der}},X^I}/X^I)$ under the identification $X^{(\lambda_1,\cdots,\lambda_{|I|})} \xrightarrow{\sim} X^I$. We note a simple Lemma:

Lemma 3.5. *Let Y be a connected, Noetherian scheme and $\mathcal{F}$ be an étale sheaf on Y. Suppose furthermore that $\mathcal{F}$ is étale locally isomorphic to a subsheaf of a constant sheaf. Then a section $s \in \Gamma(Y, \mathcal{F})$ vanishes if and only if it does so over some étale open $V \to Y$.*

Proof. One can pick finitely many étale maps $V_i \to Y$ $(i \in I)$ so that:

(a) each V_i is connected;
(b) $\mathcal{F}\big|_{V_i}$ is isomorphic to a subsheaf of a constant sheaf;
(c) the images U_i of V_i collectively cover Y.

We induct on the cardinality of I over all connected, Noetherian schemes admitting such a cover; the base case $I = \emptyset$ is trivial. The image U of $V \to Y$ must intersect some U_i. The condition (b) implies that the restriction $s_i \in \Gamma(U_i, \mathcal{F})$ vanishes. Now, let $\mathring{Y} := \bigcup_{j\neq i} U_i$. It is *not* necessarily connected. However, the fact that Y is connected shows that U_i intersects every connected component of $\mathring{Y}$. We apply the induction hypothesis to each connected component of $\mathring{Y}$ to conclude that s vanishes. $\square$

3.2.7. Our proof that each $l^{(\lambda^I)}$ vanishes now proceeds as follows:

Step 1: $l^{(0)} = 0$. Indeed, since line bundles on $\mathrm{Gr}_{G_{\mathrm{der}},X}$ are classified by the quadratic form q_{der}, we see that $\mathcal{L}$ is trivialized when pulled back along $\mathrm{Gr}_{G_{\mathrm{der}},X} \to \mathrm{Gr}_{G,X}$. On the other hand, $\mathrm{Gr}_{G_{\mathrm{der}},X}$ appears as the fiber of $\mathfrak{p}$ along the unit map $X \hookrightarrow \mathrm{Gr}_{T_1}$. Hence $l^{(0)} = 0$.

Step 2: $l^{(\lambda)} = 0$ for all $\lambda \in \Lambda_{T_1}$. Consider the section $l^{(\lambda,-\lambda)}$ of $\underline{\mathbf{P}}^{(\lambda,-\lambda)}$. It is represented by some line bundle $\mathcal{L}^{(\lambda,-\lambda)}$ over $\mathrm{Gr}_{G,X^2} \underset{\mathrm{Gr}_{T_1,X^2}}{\times} X^{(\lambda,-\lambda)}$. We know from Step 1 that the restriction of $\mathcal{L}^{(\lambda,-\lambda)}$ to the diagonal comes from the base $X^{(0)} \hookrightarrow X^{(\lambda,-\lambda)}$. Hence, over an étale

[11] Recall that for an I-family of co-characters $\lambda^{(I)} = (\lambda_1, \cdots, \lambda_{|I|})$, there is a closed immersion $X^I \hookrightarrow \mathrm{Gr}_{T_1,X^I}$ whose image we call $X^{(\lambda_1,\cdots,\lambda_{|I|})}$.

neighborhood of $X^{(0)}$, the section $l^{(\lambda,-\lambda)}$ has to vanish by the identification of $\underline{\mathbf{P}}^{(\lambda,-\lambda)}$ with $\underline{\mathbf{Pic}}(\mathrm{Gr}_{G_{\mathrm{der}},X^2}/X^2)$. We then apply Lemma 3.5 to conclude that $l^{(\lambda,-\lambda)}$ vanishes.

Now, under the identification of $\underline{\mathbf{P}}^{(\lambda,-\lambda)}$ with $\underline{\mathbf{P}}^{(\lambda)}\boxtimes\underline{\mathbf{P}}^{(-\lambda)}$ away from the diagonal, the section $l^{(\lambda,-\lambda)}$ passes to $l^{(\lambda)}\boxtimes l^{(-\lambda)}$. The fact that $l^{(\lambda,-\lambda)}=0$ now implies that $l^{(\lambda)}$ (and $l^{(-\lambda)}$) vanishes.

Step 3: $l^{(\lambda^I)}=0$ for all I-tuple λ^I. When the cardinality of I is at least 2, we may use the factorization property of $l^{(\lambda^I)}$ to see that $l^{(\lambda^I)}$ vanishes away from the union of the diagonals in $X^{(\lambda_1,\cdots,\lambda_{|I|})}$. Hence by Lemma 3.5 again we have $l^{(\lambda^I)}=0$.

This finishes the proof that (3.4) is an equivalence.

3.2.8. *Proof of* (*b*). We first recall some standard results.

Lemma 3.6. *Suppose $\widetilde{G}$ is semisimple and simply connected. Then the morphism $\mathrm{Gr}_{\widetilde{G}}\to\mathrm{Ran}(X)$ has the property that for every affine scheme $S\to\mathrm{Ran}(X)$, we have a presentation*

$$\mathrm{Gr}_{\widetilde{G}}\underset{\mathrm{Ran}(X)}{\times}S\xrightarrow{\sim}\operatorname*{colim}_i Y_i$$

where each Y_i is a scheme of finite type over S, satisfying:

(a) Y_i is proper and faithfully flat over S;

(b) The fiber $(Y_i)_s$ at every k-point $s\in S$ is connected and $\mathrm{H}^1((Y_i)_s,\mathcal{O})\cong 0$.

Proof. Since each $S\to\mathrm{Ran}(X)$ factors through some X^I, it suffices to produce such a presentation for $\mathrm{Gr}_{\widetilde{G},X^I}$. For each I-tuple $\underline{\lambda}$ of elements of Λ^+_G, we may consider the Schubert variety $\mathrm{Gr}^{\leq\underline{\lambda}}_{\widetilde{G},X^I}$ which is proper, surjective over X^I. The flatness is proved in [Zh09, §1.2] for $I=\{1,2\}$ and the general case is similar. The property (b) of its fibers is a special case of Lemma 2.6. □

Remark 3.7. Lemma 3.6(b) fails for non-semisimple groups, since Gr_G may not be ind-reduced. We do *not* know whether the flatness in part (a) holds more generally.

3.2.9. Suppose $p:X\to Y$ is a morphism of finite type schemes over k[12] such that

(a) p is proper and faithfully flat;

(b) its fiber X_y at every k-point $y\in Y$ is connected and $\mathrm{H}^1(X_y,\mathcal{O})=0$.

Lemma 3.8. *Let $\mathcal{L}$ be a line bundle on X. Under the above hypotheses on $p:X\to Y$, the following are equivalent:*

(a) $\mathcal{L}$ is trivial along the fibers of p;

(b) $p_*\mathcal{L}$ is a line bundle over Y, and the canonical map $p^*p_*\mathcal{L}\to\mathcal{L}$ is an isomorphism.

Proof. We use the formulation of the "cohomology and base change" theorem in [Va, 28.1.6]. The fiberwise triviality of $\mathcal{L}$, together with the vanishing of $\mathrm{H}^1(X_y,\mathcal{O}_{X_y})$, shows that the canonical map:

$$\mathrm{R}^1p_*\mathcal{L}\big|_y\to\mathrm{H}^1(X_y,\mathcal{L}\big|_{X_y})\tag{3.7}$$

is surjective, for any k-point $y\in Y$. Hence part (i) of *loc.cit.* applies and we see that that (3.7) is an isomorphism. Since $\mathrm{R}^1p_*\mathcal{L}$ is coherent, it must vanish. In particular, part (ii) of *loc.cit.* applies and shows that the canonical map $p_*\mathcal{L}\big|_y\to\mathrm{H}^0(X_y,\mathcal{L}\big|_{X_y})$ is surjective. Another application of part (i) then shows that $p_*\mathcal{L}$ is locally free near y of rank $h^0(X_y,\mathcal{L}\big|_{X_y})=h^0(X_y,\mathcal{O})=1$, i.e., it is a line bundle. The isomorphism $p^*p_*\mathcal{L}\xrightarrow{\sim}\mathcal{L}$ is then obvious. □

[12] Recall that k is assumed to be algebraically closed.

3.2.10. Suppose $p : \mathcal{X} \to Y$ is ind-schematic morphism, represented by morphisms $p_i : X_i \to Y$ of schemes satisfying the hypothesis of §3.2.9. Then $p^* : \mathbf{Pic}(Y) \to \mathbf{Pic}(\mathcal{X})$ has a partially defined right adjoint:

$$p_*\mathcal{L} := \lim_i (p_i)_*\mathcal{L}_i, \quad \text{while representing } \mathcal{L} \text{ by the inverse system } \mathcal{L}_i \in \mathbf{Pic}(X_i)$$

which is well defined on the full subcategory of $\mathbf{Pic}(\mathcal{X})$ consisting of line bundles which are trivial along the fibers of p, and we furthermore have an isomorphism $p^*p_*\mathcal{L} \xrightarrow{\sim} \mathcal{L}$. For any line bundle $\mathcal{M}$ from the base Y, it is also clear that $\mathcal{M} \xrightarrow{\sim} p_*p^*\mathcal{M}$. Hence p^* defines an equivalence from $\mathbf{Pic}(Y)$ to the full subcategory of $\mathbf{Pic}(\mathcal{X})$ consisting of fiberwise trivial line bundles.

3.2.11. The above discussion, together with Lemma 3.4 and 3.6, shows that $\mathfrak{p}^*$ defines an equivalence $\mathbf{Pic}(\mathrm{Gr}_{T_1}) \xrightarrow{\sim} \mathbf{Pic}_\natural(\mathrm{Gr}_G)$. To see that this upgrades to an equivalence of factorization line bundles, we simply note that the map $\mathrm{Gr}_G \underset{\mathrm{Ran}(X)}{\times} \mathrm{Gr}_G \to \mathrm{Gr}_{T_1} \underset{\mathrm{Ran}(X)}{\times} \mathrm{Gr}_{T_1}$ again satisfies the hypothesis of §3.2.10 after base change to a scheme. This finishes the proof that (3.5) is an equivalence. $\square$(Theorem 3.1)

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