

# $p$ -TORSION ÉTALE SHEAVES ON THE JACOBIAN OF A CURVE

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ABSTRACT. Suppose  $X$  is a smooth, proper, geometrically connected curve over  $\mathbb{F}_q$  with an  $\mathbb{F}_q$ -rational point  $x_0$ . For any  $\mathbb{F}_q^\times$ -character  $\sigma$  of  $\pi_1(X)$  trivial on  $x_0$ , we construct a functor  $\mathbb{L}_n^\sigma$  from the derived category of coherent sheaves on the moduli space of deformations of  $\sigma$  over the Witt ring  $W_n(\mathbb{F}_q)$  to the derived category of constructible  $W_n(\mathbb{F}_q)$ -sheaves on the Jacobian of  $X$ . The functors  $\mathbb{L}_n^\sigma$  categorify the Artin reciprocity map for geometric class field theory with  $p$ -torsion coefficients. We then give a criterion for the fully faithfulness of (an enhanced version of)  $\mathbb{L}_n^\sigma$  in terms of the Hasse–Witt matrix of  $X$ .

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## INTRODUCTION

### 0.1. Results of this paper.

0.1.1. Fix a prime  $p$ . Let  $X$  be a smooth, proper, geometrically connected curve over  $\mathbb{F}_q$ , where  $q$  is a power of  $p$ . The Artin reciprocity map of (unramified, global) geometric class field theory is a homomorphism of pro-finite groups:

$$\mathrm{Pic}(\mathbb{F}_q) \rightarrow \pi_1(X)_{\mathrm{ab}}, \quad (0.1)$$

where  $\mathrm{Pic}$  denotes the Picard scheme of  $X$ , and  $\pi_1(X)_{\mathrm{ab}}$  is the abelianization of the étale fundamental group. P. Deligne constructed the homomorphism (0.1) by a geometric argument that relates  $\ell$ -adic characters of both sides, where  $\ell$  is a prime different from  $p$ . Its key step lies in establishing an association:

$$\left\{ \begin{array}{c} \text{rank-1 étale} \\ \overline{\mathbb{Q}}_\ell\text{-local systems on } X \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \text{character } \overline{\mathbb{Q}}_\ell\text{-sheaves} \\ \text{on Pic} \end{array} \right\}. \quad (0.2)$$

The character  $\overline{\mathbb{Q}}_\ell$ -sheaf attached to a rank-1 étale local system satisfies a certain Hecke property, which ensures that its trace of Frobenius defines the correct character of  $\mathrm{Pic}(\mathbb{F}_q)$ . (See [To11] for an exposition of this approach.)

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0.1.2. In the case of  $p$ -torsion coefficients, this paper presents an alternative approach to constructing an analogue of the association (0.2). It has the additional benefit of yielding some information about deformations of  $p$ -torsion characters of  $\pi_1(X)$ . This approach is categorical in nature and is based on the Fourier–Mukai–Laumon transform.

0.1.3. Let  $W_n$  denote the length- $n$  Witt ring of  $\mathbb{F}_q$  and we fix a smooth lift  $X_n$  of the curve  $X$  to  $W_n$ . For simplicity, we further assume that  $X$  has an  $\mathbb{F}_q$ -rational point  $x_0$ , and we will only be concerned with  $W_n^\times$ -characters of  $\pi_1(X)$  trivial on  $\pi_1(x_0)$ . (Here,  $\pi_1(X)$  is defined by choosing a geometric point  $\bar{x}_0$  lying over  $x_0$ .) Let  $\text{Jac}$  denote the Jacobian scheme of  $X$  classifying a degree-0 line bundle together with a trivialization at  $x_0$ .

0.1.4. The first goal of this paper is to define a (contravariant) functor, valued in the derived category of constructible étale  $W_n$ -sheaves on  $\text{Jac}$ :

$$\mathbb{L}_n : \text{Coh}(\widetilde{\text{Jac}}_n^{\natural})^{\text{id}} \rightarrow W_n\text{-Shv}_c(\text{Jac})^{\text{op}}. \quad (0.3)$$

Here,  $\widetilde{\text{Jac}}_n^{\natural}$  is a certain scheme over  $W_n$  whose sections are identified with rank-1 étale  $W_n$ -local systems on  $X$  trivialized at  $x_0$ , with additional structure related to their deformations (see below). The left hand side means the derived category of coherent sheaves over  $\widetilde{\text{Jac}}_n^{\natural}$  of finite Tor-dimension over  $W_n$ , equipped with an arbitrary automorphism.

0.1.5. We call  $\mathbb{L}_n$  the  $p$ -torsion Artin reciprocity *functor*. It categorifies the association:

$$\left\{ \begin{array}{l} \text{rank-1 étale } W_n\text{-local systems} \\ \text{on } X \text{ trivialized at } x_0 \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{character } W_n\text{-sheaves} \\ \text{on } \text{Jac} \end{array} \right\}$$

in the following sense: given an object of the left hand side, viewed as a section  $\rho : \text{Spec}(W_n) \rightarrow \widetilde{\text{Jac}}_n^{\natural}$ , we may consider the pushforward  $\rho_*\mathcal{O}_{W_n}$  (together with the identity automorphism) as an object of  $\text{Coh}(\widetilde{\text{Jac}}_n^{\natural})^{\text{id}}$ . Then its image under  $\mathbb{L}_n$  is a character  $W_n$ -sheaf on the Jacobian satisfying the Hecke eigen-property.

0.1.6. The main result of the paper concerns the behavior of the functor  $\mathbb{L}_n$ .

**Theorem.** *The following are equivalent:*

- (a) *The functor  $\mathbb{L}_n$  is fully faithful;*
- (b) *The  $\text{Fr}_X^*$ -action on  $H^1(X; \mathcal{O}_X)$  is nilpotent.*

The  $\text{Fr}_X^*$ -action on  $H^1(X; \mathcal{O}_X)$  is known as the Hasse–Witt matrix of  $X$ . For example, when  $X$  has genus 1, condition (b) means that  $X$  is a supersingular elliptic curve.

0.1.7. We now turn to an overview of the construction of  $\mathbb{L}_n$ . Let  $\text{Jac}_n$  denote the Jacobian of the lift  $X_n$  and  $\widetilde{\text{Jac}}_n$  be its universal additive extension, which classifies  $\mathcal{L} \in \text{Jac}_n$  equipped with an integrable connection. Our construction uses the Fourier–Mukai–Laumon transform for crystalline  $\mathcal{D}$ -modules (i.e., quasi-coherent sheaves equipped with an integrable connection):

$$\Phi : \text{QCoh}(\widetilde{\text{Jac}}_n) \xrightarrow{\sim} \mathcal{D}\text{-Mod}(\text{Jac}_n). \quad (0.4)$$

Here, the two sides refer to the derived category of quasi-coherent sheaves, respectively  $\mathcal{D}$ -modules. We will verify a Frobenius compatibility statement of the functor  $\Phi$ , which asserts that the following diagram is commutative:

$$\begin{array}{ccc} \text{QCoh}(\widetilde{\text{Jac}}_n) & \xrightarrow{\Phi} & \mathcal{D}\text{-Mod}(\text{Jac}_n) \\ \downarrow \text{Ver}_* & & \downarrow \text{Fr}^* \\ \text{QCoh}(\widetilde{\text{Jac}}_n) & \xrightarrow{\Phi} & \mathcal{D}\text{-Mod}(\text{Jac}_n) \end{array} \quad (0.5)$$

Here, the Frobenius-pullback of  $\mathcal{D}$ -modules is well defined *without* a global lift of Frobenius, per an observation of P. Berthelot [Be00]. The Verschiebung endomorphism on  $\widetilde{\text{Jac}}_n$  is defined by Frobenius pullback of  $\mathcal{D}$ -modules on the curve, and we set  $\widetilde{\text{Jac}}_n^\natural$  to be its fixed-point locus.

0.1.8. The functor  $\mathbb{L}_n$  arises from taking “categorical fixed points” of  $\text{Ver}_*$  and  $\text{Fr}^*$  on both sides of (0.4). Namely, we use the  $p$ -torsion Riemann–Hilbert correspondence of Emerton–Kisin [EK04] to relate Frobenius-fixed points in  $\mathcal{D}\text{-Mod}(\text{Jac}_n)$  to constructible étale  $W_n$ -sheaves on  $\text{Jac}$ , and the Verschiebung-fixed point locus  $\widetilde{\text{Jac}}_n^\natural$  to  $W_n$ -local systems on  $X$ .

It is worth emphasizing that the process of taking “categorical fixed points” is  $\infty$ -category; namely, the analogous operation on the level of triangulated categories would yield an incorrect category. The interpretation of the Emerton–Kisin Riemann–Hilbert correspondence through categorical fixed points seems to be a fruitful idea, and we expect it to have other applications in the study of  $p$ -torsion constructible étale sheaves.

0.1.9. The present paper is motivated by certain perspectives from the geometric Langlands program in characteristic  $p$ . Indeed, for  $n = 1$ , the transformation (0.4) generalizes to a fully faithful embedding of quasi-coherent sheaves on a dense open substack of  $\text{LocSys}_r$ , the stack of rank- $r$  crystalline local systems, into  $\mathcal{D}$ -modules on the stack of rank- $r$  vector bundles, by the work of Bezrukavnikov–Braverman [BB06].

Whether or not this embedding generalizes to quasi-coherent sheaves on the entire stack  $\text{LocSys}_r$  remains speculative. However, if it does and satisfies the analogue of Frobenius compatibility (0.5), techniques of the present paper would indicate a way to study mod- $p$  (or more generally,  $p$ -torsion) automorphic sheaves via Langlands dual data.

## 0.2. Organization.

0.2.1. In the first section, we recall the  $p$ -torsion Riemann–Hilbert correspondence due to Emerton–Kisin. The key result is Proposition 1.12 (and its Corollary) which allows us to view constructible  $W_n$ -sheaves as  $\text{Fr}^*$ -fixed points of the  $\infty$ -category of crystalline  $\mathcal{D}$ -modules.

0.2.2. In the second section, we verify that the Fourier–Mukai–Laumon transform satisfies Frobenius compatibility. This fact is then used to build the main functor  $\mathbb{L}_n$ . We then show that  $\mathbb{L}_n$  de-categorifies into the Artin reciprocity map for  $W_n^\times$ -characters.

0.2.3. In the final section, we prove the main Theorem. The construction of  $\mathbb{L}_n$  makes it clear that the only source of failure of fully faithfulness lies in the passage from modules on the Verschiebung-fixed locus  $\widetilde{\text{Jac}}_n^\natural$  to modules on  $\widetilde{\text{Jac}}_n$  equipped with an isomorphism to its Verschiebung-pushforward. This can be seen as a general phenomenon that arises in comparing geometric, *vis-à-vis* categorial, fixed points. Our main result, Proposition 3.3, answers this general question when the geometric fixed-point locus is a regularly immersed affine scheme.

## 0.3. Notations.

0.3.1. We comment on our notations pertaining to category theory. The notations having to do with geometric objects will be explained at the beginning of each section.

0.3.2. In this paper, we use the theory of  $\infty$ -categories as developed by J. Lurie [Lu09] [Lu17]. We do not make model-dependent arguments, so any flavor of  $\infty$ -categories with the same formal structure suffices for our purpose.

0.3.3. For an associative algebra  $A$  (in the classical sense), we write  $A\text{-Mod}$  for the  $\infty$ -category of chain complexes of  $A$ -modules. When we work with the abelian subcategory, we denote it by  $A\text{-Mod}^\heartsuit$ , understood as the heart of the usual  $t$ -structure on  $A\text{-Mod}$ . In line with this notation, we often write “ $A$ -module” to mean an object of  $A\text{-Mod}$ , i.e., a complex of  $A$ -modules.

0.3.4. The same convention applies to the  $\infty$ -categories  $\text{QCoh}(Y)$ ,  $\mathcal{D}\text{-Mod}(Y)$ , etc., for a scheme  $Y$ . By default, all functors are derived. However, sometimes we emphasize their derived nature using notations such as  $Li^*$ ,  $Rf_*$ .

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## 1. UNIT $F$ -CRYSTALS

In this section, we first recall the  $p$ -torsion Riemann–Hilbert correspondence due to Emerton–Kisin [EK04]. It asserts that the derived category of  $p$ -torsion étale sheaves is (anti-)equivalent to that of arithmetic  $\mathcal{D}$ -modules  $\mathcal{M}$  equipped with an isomorphism to its Frobenius pullback  $\psi_{\mathcal{M}} : \mathcal{M} \xrightarrow{\sim} \text{Fr}^* \mathcal{M}$ , satisfying some finiteness conditions. We then show that the datum of  $\psi_{\mathcal{M}}$  can be expressed on the level of chain complexes, provided that one works with stable  $\infty$ -categories.

### 1.1. The Riemann–Hilbert correspondence.

1.1.1. In this section, we fix a perfect field  $k$  containing  $\mathbb{F}_q$ . Let  $W_n(k)$  (resp.  $W_n(\mathbb{F}_q)$ ) denote the ring of length- $n$  Witt vectors over  $k$  (resp.  $\mathbb{F}_q$ ); it is equipped with a canonical lift of the  $q$ th power Frobenius, denoted by  $\text{Fr}_{W_n(k)}$ . Suppose  $Y$  is a *smooth* scheme over  $W_n(k)$ . Let  $Y_0$  denote the base change of  $Y$  to  $k$ .

1.1.2. Denote by  $W_n(\mathbb{F}_q)\text{-Shv}(Y_0)$  the  $\infty$ -category of étale  $W_n(\mathbb{F}_q)$ -sheaves on  $Y_0$ . More precisely, it is the  $\infty$ -category of  $W_n(\mathbb{F}_q)\text{-Mod}$ -valued étale sheaves of [GL19, Definition 2.2.1.2]. It contains as full subcategory:

$$W_n(\mathbb{F}_q)\text{-Shv}_c(Y_0) \subset W_n(\mathbb{F}_q)\text{-Shv}(Y_0)$$

the  $\infty$ -category of constructible étale  $W_n(\mathbb{F}_q)$ -sheaves, which consists of objects  $\mathcal{F}$  of finite Tor-dimension over  $W_n(\mathbb{F}_q)$  and such that each  $H^i(\mathcal{F})$  is constructible (see [De77, §2, Proposition-définition 4.6]).<sup>1</sup> The Riemann–Hilbert correspondence of Emerton–Kisin [EK04] expresses  $W_n(\mathbb{F}_q)\text{-Shv}_c(Y_0)$  in terms of *unit  $F$ -crystals* on  $Y$ . The goal of this subsection is to review these objects and state a form of the Riemann–Hilbert correspondence.

1.1.3. For an integer  $\nu \geq 0$ , let  $\mathcal{D}_Y^{(\nu)}$  denote the ring of  $W_n(k)$ -linear differential operators on  $Y$  with level- $\nu$  divided powers; it can be thought of as the  $\mathcal{O}_Y$ -algebra generated by differential operators of the form  $\partial^{p^k}/p^k!$  for  $k \leq \nu$ . More formally, the ring  $\mathcal{D}_Y^{(\nu)}$  is defined to be the topological dual of functions on  $(Y \times_{W_n(k)} Y)^{(\nu\text{-PD})}_{\widehat{Y}}$ , the level- $\nu$  divided power neighborhood of

<sup>1</sup>The homotopy category of  $W_n(\mathbb{F}_q)\text{-Shv}_c(Y)$  is denoted by  $D_{\text{ctf}}^b(X_{\text{ét}}, W_n(\mathbb{F}_q))$  in [EK04].

the closed immersion  $\Delta : Y \hookrightarrow Y \times_{W_n(k)} Y$  (see [Be96, §2.2.1]). Therefore, a quasi-coherent  $\mathcal{D}_Y^{(\nu)}$ -module is equivalent to a quasi-coherent  $\mathcal{O}_Y$ -module equivariant with respect to the action of the divided power infinitesimal groupoid:<sup>2</sup>

$$\cdots \rightrightarrows (Y \times_{W_n(k)} Y)_{\hat{Y}}^{(\nu\text{-PD})} \rightrightarrows Y.$$

1.1.4. There is a sequence of (non-injective) morphisms of filtered algebras:

$$\mathcal{D}_Y^{(0)} \rightarrow \mathcal{D}_Y^{(1)} \rightarrow \cdots \rightarrow \mathcal{D}_Y := \operatorname{colim}_{\nu} \mathcal{D}_Y^{(\nu)}.$$

In particular, there are forgetful functors in the opposite direction:

$$\mathcal{D}_Y\text{-Mod} \rightarrow \cdots \rightarrow \mathcal{D}_Y^{(1)}\text{-Mod} \rightarrow \mathcal{D}_Y^{(0)}\text{-Mod}.$$

Objects of  $\mathcal{D}_Y^{(0)}$ -Mod are usually called *crystalline*  $\mathcal{D}$ -modules.<sup>3</sup> They are precisely quasi-coherent  $\mathcal{O}_Y$ -modules equipped with an integrable connection relative to  $W_n(k)$ .

1.1.5. An important observation due to P. Berthelot [Be00, §2] is that there is a canonically defined  $W_n(\mathbb{F}_q)$ -linear endofunctor  $\operatorname{Fr}_Y^*$  on  $\mathcal{D}_Y^{(\nu)}$ -Mod<sup>∇</sup>, which can be constructed by choosing any local lift  $\operatorname{Fr}_Y$  of the  $q$ th power Frobenius and showing that the resulting  $\mathcal{D}_Y^{(\nu)}$ -module is canonically independent of the lift. Furthermore,  $\operatorname{Fr}_Y^*$  enhances to a functor (denoted by the same name):

$$\operatorname{Fr}_Y^* : \mathcal{D}_Y^{(\nu)}\text{-Mod}^{\heartsuit} \rightarrow \mathcal{D}_Y^{(\nu+1)}\text{-Mod}^{\heartsuit}. \quad (1.1)$$

**Remark 1.1.** In fact, the functor (1.1) is an equivalence of categories, by Théorème 2.3.6 of *op.cit.*, but we shall not use this fact.

1.1.6. We write  $\mathcal{D}_{F,Y}^{(\nu)}\text{-Mod}^{\heartsuit}$  for the abelian category of objects  $\mathcal{M} \in \mathcal{D}_Y^{(\nu)}\text{-Mod}^{\heartsuit}$  together with a morphism  $\psi_{\mathcal{M}} : \operatorname{Fr}_Y^* \mathcal{M} \rightarrow \mathcal{M}$ . The datum of  $\psi_{\mathcal{M}}$  can alternatively be described as an enhancement of the  $\mathcal{D}_Y^{(\nu)}$ -action to an action of:

$$\mathcal{D}_{F,Y}^{(\nu)} := \bigoplus_{r \geq 0} (\operatorname{Fr}_Y^*)^r \mathcal{D}_Y^{(\nu)},$$

for a canonically defined algebra structure on  $\mathcal{D}_{F,Y}^{(\nu)}$  (see [EK04, §13.3]). We call  $\mathcal{D}_{F,Y}^{(\nu)}\text{-Mod}^{\heartsuit}$  the abelian category of *F-crystals of level  $\nu$* . It contains enough injective objects and we may write  $D^+(\mathcal{D}_{F,Y}^{(\nu)}\text{-Mod}^{\heartsuit})$  for its (bounded below) derived  $\infty$ -category.

1.1.7. We introduce two of its full subcategories:

$$\mathcal{D}_{F,Y}^{(\nu)}\text{-Mod}_{\text{lf-g-u}}^{\heartsuit} \subset \mathcal{D}_{F,Y}^{(\nu)}\text{-Mod}_{\text{u}}^{\heartsuit} \subset \mathcal{D}_{F,Y}^{(\nu)}\text{-Mod}^{\heartsuit}.$$

They are defined by successively imposing the following conditions:

- (a)  $\mathcal{D}_{F,Y}^{(\nu)}\text{-Mod}_{\text{u}}^{\heartsuit}$  — where  $\psi_{\mathcal{M}}$  is an isomorphism; these are called *unit F-crystals*. We note that by (1.1), the  $\mathcal{D}_Y^{(\nu)}$ -module structure is promoted to a  $\mathcal{D}_Y$ -module structure. Hence the notion of *unit F-crystals* is independent of the level  $\nu$ , i.e. the forgetful functors:

$$\mathcal{D}_{F,Y}\text{-Mod}_{\text{u}}^{\heartsuit} \rightarrow \cdots \rightarrow \mathcal{D}_{F,Y}^{(1)}\text{-Mod}_{\text{u}}^{\heartsuit} \rightarrow \mathcal{D}_{F,Y}^{(0)}\text{-Mod}_{\text{u}}^{\heartsuit}$$

are equivalences of abelian categories.

<sup>2</sup>This equivariance datum is called  *$\nu$ -PD-stratification* in *loc.cit.*

<sup>3</sup>We change the notation from the introduction, where  $\mathcal{D}_Y^{(0)}$ -modules was denoted by  $\mathcal{D}_Y\text{-Mod}$ .

- (b)  $\mathcal{D}_{F,Y}^{(\nu)}\text{-Mod}_{\text{lf-g-u}}^{\heartsuit}$  — where  $\mathcal{M}$  further contains an  $\mathcal{O}_Y$ -coherent submodule  $\mathcal{M}_0$  such that the action map  $\mathcal{D}_{F,Y}^{(\nu)} \otimes_{\mathcal{O}_Y} \mathcal{M}_0 \rightarrow \mathcal{M}$  is surjective.

1.1.8. Write  $D_{\text{lf-g-u},\circ}^b(\mathcal{D}_{F,Y}^{(\nu)}\text{-Mod}^{\heartsuit})$  for the full  $\infty$ -subcategory of  $D^+(\mathcal{D}_{F,Y}^{(\nu)}\text{-Mod}^{\heartsuit})$  consisting of complexes  $\mathcal{M}$  satisfying the following conditions:

- (a)  $\mathcal{M}$  is of finite Tor-dimension as a complex of  $\mathcal{O}_Y$ -modules;  
 (b)  $H^i(\mathcal{M})$  belongs to  $\mathcal{D}_{F,Y}^{(\nu)}\text{-Mod}_{\text{lf-g-u}}^{\heartsuit}$  for each  $i$ .

The functor (1.1) can be promoted to  $\text{Fr}_Y^* : \mathcal{D}_{F,Y}^{(\nu)}\text{-Mod}^{\heartsuit} \rightarrow \mathcal{D}_{F,Y}^{(\nu+1)}\text{-Mod}^{\heartsuit}$  and thus on the derived  $\infty$ -category. In particular, we see that  $D_{\text{lf-g-u},\circ}^b(\mathcal{D}_{F,Y}^{(\nu)}\text{-Mod}^{\heartsuit})$  is also independent of the level  $\nu$ , i.e., the following forgetful functors:

$$D_{\text{lf-g-u},\circ}^b(\mathcal{D}_{F,Y}\text{-Mod}^{\heartsuit}) \rightarrow \cdots \rightarrow D_{\text{lf-g-u},\circ}^b(\mathcal{D}_{F,Y}^{(1)}\text{-Mod}^{\heartsuit}) \rightarrow D_{\text{lf-g-u},\circ}^b(\mathcal{D}_{F,Y}^{(0)}\text{-Mod}^{\heartsuit}) \quad (1.2)$$

are equivalences.

1.1.9. Define a functor of  $W_n(\mathbb{F}_q)$ -linear  $\infty$ -categories (here, we take  $\nu$  to be infinite):

$$\text{Sol} : D^b(\mathcal{D}_{F,Y}\text{-Mod}^{\heartsuit}) \rightarrow W_n(\mathbb{F}_q)\text{-Shv}^+(Y_0),$$

by sending  $\mathcal{M}$  to:

$$\text{Sol}(\mathcal{M}) := R\text{Hom}_{\mathcal{D}_{F,Y}}(\mathcal{M}_{\text{ét}}, \mathcal{O}_Y)[\dim(Y)],$$

where  $\mathcal{M}_{\text{ét}}$  is the étale  $\mathcal{D}_{F,Y}$ -module associated to  $\mathcal{M}$ .

**Theorem 1.2** (Emerton–Kisin). *The restriction of  $\text{Sol}$  to  $D_{\text{lf-g-u},\circ}^b(\mathcal{D}_{F,Y}\text{-Mod}^{\heartsuit})$  defines an anti-equivalence of  $W_n(\mathbb{F}_q)$ -linear  $\infty$ -categories:*

$$D_{\text{lf-g-u},\circ}^b(\mathcal{D}_{F,Y}\text{-Mod}^{\heartsuit})^{\text{op}} \xrightarrow{\sim} W_n(\mathbb{F}_q)\text{-Shv}_c(Y_0). \quad (1.3)$$

*Proof.* It suffices to check that  $\text{Sol}$  becomes an equivalence after passing to the homotopy categories. For  $q = p$ , this is the main theorem of [EK04], but their proof also applies to the case where  $q$  is a power of  $p$ .  $\square$

**Remark 1.3.** The equivalences (1.2) show that the same result holds for  $\mathcal{D}_{F,Y}^{(\nu)}$ -modules of any level  $\nu \geq 0$ .

## 1.2. Katz’s equivalence.

1.2.1. As in the characteristic-zero case, (1.3) specializes to an equivalence of 1-categories corresponding to *lisse* sheaves. The latter equivalence is essentially due to N. Katz [Ka73]. We will need a generalization of this special case where we allow deformations over  $W_n(\mathbb{F}_q)$ . The result is likely well known, but we could not find a suitable reference.

**Lemma 1.4.** *Suppose  $S = \text{Spec}(A)$  is a finite  $W_n(\mathbb{F}_q)$ -scheme. Then the following categories are equivalent functorially in  $S$ :*

- (a) *étale  $A$ -sheaves on  $Y$  which are locally isomorphic to  $\underline{A}^{\oplus r}$ ;*  
 (b) *rank- $r$  vector bundles  $\mathcal{E}$  on  $S \times_{W_n(\mathbb{F}_q)} Y$  equipped with an integrable connection along  $Y$  relative to  $W_n(k)$ , and an isomorphism  $\text{Fr}_Y^* \mathcal{E} \xrightarrow{\sim} \mathcal{E}$  as  $\mathcal{D}^{(0)}$ -modules.*

Here, the symbol  $\mathcal{D}^{(0)}$  stands for the level-0 differential operators on  $S \times_{W_n(\mathbb{F}_q)} Y$  relative to

$$S \times_{W_n(\mathbb{F}_q)} W_n(k).$$

1.2.2. The case  $A = W_n(\mathbb{F}_q)$  is essentially Proposition 4.1.1 of *loc.cit.*, the only difference being that Katz's equivalence uses a global lift of the Frobenius on  $Y$  whereas we use  $\mathcal{D}^{(0)}$ -modules and their canonical Frobenius pullbacks. We will explain how the proof of the cited Proposition can be adapted to yield a proof of Lemma 1.4.

1.2.3. One first defines a functor  $F$  from (a) to (b). Let  $\mathcal{F}$  be an object of the category (a). By passing to an étale cover of  $Y$ , we may assume that  $\mathcal{F}$  is trivialized. We will then construct an object of (b) while remembering its descent datum. As a quasi-coherent  $\mathcal{O}_Y$ -module, we set:<sup>4</sup>

$$\mathcal{E} := \mathcal{F} \otimes_{W_n(\mathbb{F}_q)} \mathcal{O}_Y.$$

The integrable connection and Frobenius structure on  $\mathcal{E}$  are both defined by actions on the  $\mathcal{O}_Y$ -factor. Finally, the  $A$ -module structure on  $\mathcal{F}$  induces an  $A$ -action on  $\mathcal{E}$ . This turns  $\mathcal{E}$  into a quasi-coherent sheaf on  $S \times_{W_n(\mathbb{F}_q)} Y$ , which is locally free of rank  $r$ .

1.2.4. To show that this functor is an equivalence is a local question on  $Y$ . Hence we may assume that  $Y$  is equipped with a lift  $\mathrm{Fr}_Y$  of the  $q$ th power Frobenius endomorphism on its special fiber. We then define another category:

(b') vector bundles  $\mathcal{E}$  on  $S \times_{W_n(\mathbb{F}_q)} Y$  equipped with an isomorphism  $\psi : \mathrm{Fr}_Y^* \mathcal{E} \xrightarrow{\sim} \mathcal{E}$ .

**Claim 1.5.** The forgetful functor from (b) to (b') is an equivalence.

*Proof.* Recall that for an integer  $N \gg 0$  (depending only on  $Y$ ) and any  $\mathcal{O}_Y$ -module  $\mathcal{M}$ , the quasi-coherent sheaf  $(\mathrm{Fr}_Y^*)^N \mathcal{M}$  acquires a *canonical* connection  $\nabla_{\mathrm{can}}$ , defined functorially in  $\mathcal{M}$ . Furthermore, if  $\mathcal{M}$  already admits a connection, then  $\nabla_{\mathrm{can}}$  identifies with its pullback. This shows that an object  $\mathcal{E}$  in (b') acquires a connection, making the isomorphism:

$$\psi^N : (\mathrm{Fr}_Y^*)^N \mathcal{E} \xrightarrow{\sim} \mathcal{E}$$

$\mathcal{D}^{(0)}$ -equivariant. To show that this connection upgrades  $\mathcal{E}$  to an object of (b), we must show that the structural morphism  $\psi$  is itself  $\mathcal{D}^{(0)}$ -equivariant. However, this follows from considering the following commutative diagram:

$$\begin{array}{ccc} (\mathrm{Fr}_Y^*)^{N+1} \mathcal{E} & \xrightarrow{(\mathrm{Fr}_Y^*)^N \psi} & (\mathrm{Fr}_Y^*)^N \mathcal{E} \\ \downarrow \mathrm{Fr}_Y^* \psi^N & & \downarrow \psi^N \\ \mathrm{Fr}_Y^* \mathcal{E} & \xrightarrow{\psi} & \mathcal{E} \end{array}$$

where all arrows besides  $\psi$  are already known to be  $\mathcal{D}^{(0)}$ -equivariant. □

1.2.5. Thus we have reduced to showing the functor  $F'$  from (a) to (b'), given by composing  $F$  with the forgetful functor, is an equivalence. It admits an *a priori* partially defined right adjoint  $(F')^R$ , given by sending  $(\mathcal{E}, \psi)$  (regarded as a quasi-coherent  $\mathcal{O}_Y$ -module with Frobenius structure  $\psi$ ) to the fixed points of the Frobenius-twisted linear morphism:

$$\varphi : \mathcal{E} \hookrightarrow \mathrm{Fr}_Y^* \mathcal{E} \xrightarrow{\psi} \mathcal{E}.$$

We denote the resulting étale  $W_n(\mathbb{F}_q)$ -module by  $\mathcal{E}^{\varphi\text{-fixed}}$ ; the  $A$ -action on  $\mathcal{E}$  induces one on  $\mathcal{E}^{\varphi\text{-fixed}}$ , making it an étale  $A$ -sheaf. This functor is well-defined precisely when  $\mathcal{E}^{\varphi\text{-fixed}}$  is locally isomorphic to a product of the constant  $A$ -sheaf.

<sup>4</sup>We note that since  $k$  is a perfect field,  $W_n(k)$  is flat over  $W_n(\mathbb{F}_q)$ . Thus  $\mathcal{O}_Y$  is also flat over  $W_n(\mathbb{F}_q)$  and the tensor product can be safely taken in the classical sense.

1.2.6. The following claim shows that  $(F')^R$  is well-defined on the essential image of  $F'$  and that  $F'$  is fully faithful.

**Claim 1.6.** For any finite, étale  $W_n(\mathbb{F}_q)$ -sheaf  $\mathcal{F}$ , the following natural map is bijective:

$$\mathcal{F} \rightarrow (\mathcal{F} \otimes_{W_n(\mathbb{F}_q)} \mathcal{O}_Y)^{\varphi\text{-fixed}}.$$

*Proof.* By the classification of finite modules over a PID (applied to  $\mathcal{F}$  viewed as a  $W(\mathbb{F}_q)$ -module), we obtain an isomorphism  $\mathcal{F} \xrightarrow{\sim} \bigoplus_j \mathcal{F}_j$  where  $\mathcal{F}_j$  is a *free*  $W_j(\mathbb{F}_q)$ -module for  $1 \leq j \leq n$ . Therefore, we have:

$$\mathcal{F} \otimes_{W_n(\mathbb{F}_q)} \mathcal{O}_Y \xrightarrow{\sim} \bigoplus_{1 \leq j \leq n} \mathcal{F}_j \otimes_{W_j(\mathbb{F}_q)} \mathcal{O}_{Y_j}.$$

Here,  $Y_j$  denotes the base change of  $Y$  to  $W_j(\mathbb{F}_q)$ . The result follows, since  $W_j(\mathbb{F}_q)$  identifies with the  $\varphi$ -fixed elements of  $\mathcal{O}_{Y_j}$ .  $\square$

1.2.7. Finally, we show that  $F'$  is essentially surjective. Given any  $(\mathcal{E}, \psi)$  in (b'), this amounts to the following problem:

- Étale locally on  $Y$ , construct a basis  $\underline{h}$  of  $\mathcal{E}$  (over  $S \times_{W_n(\mathbb{F}_q)} Y$ ) such that  $\varphi$  fixes  $\underline{h}$ .

The argument is then essentially copying Katz's proof. We proceed by induction on  $n \geq 1$ .

1.2.8. For  $n = 1$ , the vector bundle  $\mathcal{E}$  is *also* locally free as an  $\mathcal{O}_Y$ -module. By *loc.cit.*, we may functorially assign a finite, locally constant étale  $\mathbb{F}_q$ -sheaf  $\mathcal{F}$  on  $Y$  together with an isomorphism  $\mathcal{F} \otimes_{\mathbb{F}_q} \mathcal{O}_Y \xrightarrow{\sim} \mathcal{E}$ . The  $A$ -action on  $\mathcal{E}$  is inherited by  $\mathcal{F}$ , making the above isomorphism  $A$ -equivariant. The hypothesis that  $\mathcal{E}$  is locally free on  $S \times_{\mathbb{F}_q} Y$  then shows that  $\mathcal{F}$  is locally constant as an  $A$ -module. This gives a  $\varphi$ -fixed basis of  $\mathcal{E}$ .<sup>5</sup>

1.2.9. Assuming that a  $\varphi$ -fixed basis exists for  $n - 1$ , we will lift it to a basis  $\underline{h}$  of  $\mathcal{E}$  over  $S \times_{W_n(\mathbb{F}_q)} Y$ ; it is not necessarily fixed by  $\varphi$ . In order to adjust it into a  $\varphi$ -fixed basis, we apply the argument in p.145 of *loc.cit.*. It reduces the problem to solving  $r^2$  equations in  $A_1 \otimes_{\mathbb{F}_q} \mathcal{O}_Y$  (where  $A_1$  is the reduction of  $A \bmod p$ ) of the following type, with  $\delta$ 's being given:

$$\varphi(e) + \delta = e. \tag{1.4}$$

Here,  $\varphi$  denotes the Frobenius action on the  $\mathcal{O}_{Y_1}$ -factor. We write  $\delta = \sum_i a_i \otimes f_i$ . The equation (1.4) then reduces to finding solutions  $x_i \in \mathcal{O}_{Y_1}$  to the equations  $x_i^q + f_i = x_i$  for all  $i$ . Indeed, having the  $x_i$ 's, we may then set  $e := \sum_i a_i \otimes x_i$ . These Artin–Schreier equations are simultaneously solved over a finite étale cover of  $Y_n$ .  $\square$ (Lemma 1.4)

1.2.10. The particular case of Lemma 1.4 where  $A = W_n(\mathbb{F}_q)$  is captured by the Emerton–Kisin Riemann–Hilbert correspondence (1.3) in the following manner. We let  $\text{LocSys}(Y)$  and  $\text{Conn}(Y)^{\text{Fr}^*}$  denote the two categories appearing in Lemma 1.4 for  $A = W_n(\mathbb{F}_q)$ . Then the following diagram commutes:

<sup>5</sup>In fact, it proves the essential surjectivity directly.

$$\begin{array}{ccc}
 \mathrm{Conn}(Y)^{\mathrm{Fr}^*} & \xrightarrow{\text{Lemma 1.4}} & \mathrm{LocSys}(Y) \\
 \downarrow \mathbb{D} & & \downarrow \\
 (\mathrm{Conn}(Y)^{\mathrm{Fr}^*})^{\mathrm{op}} & & \\
 \downarrow & & \downarrow \\
 D_{\mathrm{lf-g-u},\circ}^b(\mathcal{D}_{F,Y}^{(0)}\text{-Mod}^\heartsuit)^{\mathrm{op}} & \xrightarrow{\mathrm{Sol}[-\dim(Y)]} & W_n(\mathbb{F}_q)\text{-Shv}_c(Y)
 \end{array}$$

Here, the unlabeled vertical arrows are the tautological inclusions; the functor  $\mathbb{D}$  is the operation of taking the monoidal dual  $\mathcal{E} \rightsquigarrow \mathcal{E}^\vee$  with its induced connection and Frobenius-structure.

**Remark 1.7.** The duality functor  $\mathbb{D}$  cannot be extended to  $D_{\mathrm{lf-g-u},\circ}^b(\mathcal{D}_{F,Y}\text{-Mod}^\heartsuit)$ , so we do not obtain a covariant Riemann–Hilbert equivalence. However, a version of covariant Riemann–Hilbert equivalence *does* exist. It appeals to a different notion than  $F$ -crystals, and is the subject of the work of Bhatt–Lurie [BL17].

### 1.3. Categorical fixed points.

1.3.1. Let  $\mathcal{C}$  be a stable  $\infty$ -category together an endomorphism  $F : \mathcal{C} \rightarrow \mathcal{C}$ . The *fixed point  $\infty$ -category*  $\mathcal{C}^F$  is defined as the equalizer of  $F$  with  $\mathrm{id}_{\mathcal{C}}$ . Equivalently,  $\mathcal{C}^F$  is a fiber product of stable  $\infty$ -categories:

$$\begin{array}{ccc}
 \mathcal{C}^F & \longrightarrow & \mathcal{C} \\
 \downarrow & & \downarrow \Delta \\
 \mathcal{C} & \xrightarrow{(F,\mathrm{id})} & \mathcal{C} \times \mathcal{C}
 \end{array}$$

An object of  $\mathcal{C}^F$  is an object  $c \in \mathcal{C}$  together with an isomorphism  $Fc \xrightarrow{\sim} c$ .

1.3.2. More generally, one can define a lax version  $\mathcal{C}^{F \rightarrow \mathrm{id}}$  of objects  $c \in \mathcal{C}$  together with a morphism  $Fc \rightarrow c$ , so that  $\mathcal{C}^F$  is the full  $\infty$ -subcategory of  $\mathcal{C}^{F \rightarrow \mathrm{id}}$  where the structural morphism is an isomorphism. Formally,  $\mathcal{C}^{F \rightarrow \mathrm{id}}$  is the fiber product:

$$\begin{array}{ccc}
 \mathcal{C}^{F \rightarrow \mathrm{id}} & \longrightarrow & \mathcal{C}^{\Delta^1} \\
 \downarrow & & \downarrow \mathrm{can} \\
 \mathcal{C} & \xrightarrow{(F,\mathrm{id})} & \mathcal{C} \times \mathcal{C}
 \end{array}$$

where  $\mathcal{C}^{\Delta^1} := \mathrm{Funct}(\Delta^1, \mathcal{C})$  is the functor  $\infty$ -category from the interval, and the canonical map sends  $d \rightarrow c$  to  $(d, c)$ .

**Remark 1.8.** The  $\infty$ -category  $\mathcal{C}^{F \rightarrow \mathrm{id}}$  has an obvious variant  $\mathcal{C}^{\mathrm{id} \rightarrow F}$  consisting of objects  $c \in \mathcal{C}$  together with a morphism  $c \rightarrow Fc$ .

1.3.3. The following fact shows how to calculate Hom-spaces in  $\mathcal{C}^{F \rightarrow \mathrm{id}}$  (hence also in  $\mathcal{C}^F$ ).

**Lemma 1.9.** *Given objects  $Fc_1 \xrightarrow{\psi_1} c_1, Fc_2 \xrightarrow{\psi_2} c_2$  in  $\mathcal{C}^{F \rightarrow \mathrm{id}}$ , their Hom-space in  $\mathcal{C}^{F \rightarrow \mathrm{id}}$  is the equalizer of:*

$$\mathrm{Hom}_{\mathcal{C}}(c_1, c_2) \begin{array}{c} \xrightarrow{\psi_2 \circ F(-)} \\ \xrightarrow[(-) \circ \psi_1]{} \end{array} \mathrm{Hom}_{\mathcal{C}}(Fc_1, c_2).$$

*Proof.* The Hom-space in a limit of  $\infty$ -categories is the limit of Hom-spaces. Therefore, the problem reduces to showing that Hom-spaces in  $\mathcal{C}^{\Delta^1}$  identify with the fiber product:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}^{\Delta^1}}(d_1 \rightarrow c_1, d_2 \rightarrow c_2) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(d_1, d_2) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{C}}(c_1, c_2) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(d_1, c_2) \end{array}$$

This is in turn a particular case of the calculation of natural transformations as limit over the twisted arrow category.  $\square$

1.3.4. Suppose  $\mathcal{C}$  is equipped with a  $t$ -structure and  $F$  is  $t$ -exact. Then  $\mathcal{C}^{F \rightarrow \mathrm{id}}$  inherits a  $t$ -structure, for which the forgetful functor to  $\mathcal{C}$  is  $t$ -exact.

**Lemma 1.10.** *If  $\mathcal{C}$  is right (resp. left) complete with respect to the  $t$ -structure, then the same holds for  $\mathcal{C}^{F \rightarrow \mathrm{id}}$ .*

*Proof.* Right (resp. left) completeness of  $\mathcal{C}$  means that the canonical functor  $\mathcal{C} \rightarrow \lim_{n \rightarrow \infty} \mathcal{C}^{\leq n}$  (resp.  $\mathcal{C} \rightarrow \lim_{n \rightarrow -\infty} \mathcal{C}^{\geq n}$ ), where the connecting functors  $\mathcal{C}^{\leq n+1} \rightarrow \mathcal{C}^{\leq n}$  (resp.  $\mathcal{C}^{\geq n-1} \rightarrow \mathcal{C}^{\geq n}$ ) are given by truncations  $\tau^{\leq n}$  (resp.  $\tau^{\geq n}$ ), is an equivalence. Since  $(\mathcal{C}^{F \rightarrow \mathrm{id}})^{\leq n}$  is identified with the fiber product:

$$\begin{array}{ccc} (\mathcal{C}^{F \rightarrow \mathrm{id}})^{\leq n} & \longrightarrow & (\mathcal{C}^{\leq n})^{\Delta^1} \\ \downarrow & & \downarrow \mathrm{can} \\ \mathcal{C}^{\leq n} & \xrightarrow{(F, \mathrm{id})} & \mathcal{C}^{\leq n} \times \mathcal{C}^{\leq n} \end{array}$$

in a way that is compatible with the truncation functors, we see that  $\mathcal{C}^{F \rightarrow \mathrm{id}}$  is identified with  $\lim_{n \rightarrow \infty} (\mathcal{C}^{F \rightarrow \mathrm{id}})^{\leq n}$ . Under the appropriate hypotheses, the same argument yields the left-completeness and the corresponding statements for  $\mathcal{C}^F$ .  $\square$

#### 1.4. $F$ -crystals as categorial fixed points.

1.4.1. We return to the setting of §1.1. Note that the functor (1.1) enhances to a  $W_n(\mathbb{F}_q)$ -linear functor of  $\infty$ -categories:

$$\mathrm{Fr}_Y^* : \mathcal{D}_Y^{(\nu)}\text{-Mod} \rightarrow \mathcal{D}_Y^{(\nu+1)}\text{-Mod}. \quad (1.5)$$

Its composition with the forgetful functor to  $\mathcal{D}_Y^{(\nu)}\text{-Mod}$  will still be denoted by  $\mathrm{Fr}_Y^*$ . Let  $\mathcal{D}_Y^{(\nu)}\text{-Mod}^{\mathrm{Fr}^*}$  (resp.  $\mathcal{D}_Y^{(\nu)}\text{-Mod}^{\mathrm{Fr}^* \rightarrow \mathrm{id}}$ ) denote the (resp. lax)  $\mathrm{Fr}_Y^*$ -fixed point  $\infty$ -category of  $\mathcal{D}_Y^{(\nu)}\text{-Mod}$ . We note that the equivalence (1.5) shows that  $\mathcal{D}_Y^{(\nu)}\text{-Mod}^{\mathrm{Fr}^*}$  is independent of the level  $\nu$ , i.e., the forgetful functors:

$$\mathcal{D}_Y\text{-Mod}^{\mathrm{Fr}^*} \rightarrow \dots \rightarrow \mathcal{D}_Y^{(1)}\text{-Mod}^{\mathrm{Fr}^*} \rightarrow \mathcal{D}_Y^{(0)}\text{-Mod}^{\mathrm{Fr}^*}$$

are equivalences, with inverses given by  $\mathrm{Fr}_Y^*$ .

1.4.2. We note that  $\mathrm{Fr}_Y^*$  is  $t$ -exact with respect to the tautological  $t$ -structure on  $\mathcal{D}_Y^{(\nu)}\text{-Mod}$ ; indeed, this is because  $Y$  is smooth, so any local lift of the Frobenius map is flat. Thus  $\mathcal{D}_Y^{(\nu)}\text{-Mod}^{\mathrm{Fr}^* \rightarrow \mathrm{id}}$  inherits a  $t$ -structure, for which the forgetful functor to  $\mathcal{D}_Y^{(\nu)}\text{-Mod}$  is  $t$ -exact. The following is immediate:

**Lemma 1.11.** *The heart  $(\mathcal{D}_Y^{(\nu)}\text{-Mod}^{\mathrm{Fr}^* \rightarrow \mathrm{id}})^{\heartsuit}$  identifies with  $\mathcal{D}_{F,Y}^{(\nu)}\text{-Mod}^{\heartsuit}$ .*  $\square$

Furthermore, in order to check that an object of  $\mathcal{D}_Y^{(\nu)}\text{-Mod}^{\mathrm{Fr}^* \rightarrow \mathrm{id}}$  belongs to the full  $\infty$ -subcategory  $\mathcal{D}_Y^{(\nu)}\text{-Mod}^{\mathrm{Fr}^*}$ , it suffices to check that its cohomology objects belong to  $\mathcal{D}_{F,Y}^{(\nu)}\text{-Mod}_u^{\heartsuit}$ , i.e., are *unit  $F$ -crystals*.

1.4.3. It follows from Lemma 1.11 that we have a canonically defined,  $t$ -exact functor:

$$D^+(\mathcal{D}_{F,Y}^{(\nu)}\text{-Mod}^{\heartsuit}) \rightarrow (\mathcal{D}_Y^{(\nu)}\text{-Mod}^{\text{Fr}^* \rightarrow \text{id}})^+. \quad (1.6)$$

We specify the full  $\infty$ -subcategory  $(\mathcal{D}_Y^{(\nu)}\text{-Mod}^{\text{Fr}^*})_{\text{lfg-u},\circ} \subset (\mathcal{D}_Y^{(\nu)}\text{-Mod}^{\text{Fr}^* \rightarrow \text{id}})^+$  as consisting of objects of finite Tor-dimension over  $\mathcal{O}_Y$ , whose cohomology groups belong to  $\mathcal{D}_{F,Y}^{(\nu)}\text{-Mod}_{\text{lfg-u}}^{\heartsuit}$ . In particular, the structural morphism of objects  $\text{Fr}_Y^* \mathcal{M} \rightarrow \mathcal{M}$  in this full subcategory are isomorphisms. The same argument as in §1.1.5 shows that the  $\infty$ -category  $(\mathcal{D}_Y^{(\nu)}\text{-Mod}^{\text{Fr}^*})_{\text{lfg-u},\circ}$  is defined independently of the level  $\nu$ .

1.4.4. The following Proposition will be proved in the next subsection:

**Proposition 1.12.** *The functor (1.6) is an equivalence of  $\infty$ -categories.*

**Corollary 1.13.** *There is a canonical (anti-)equivalence of  $W_n(\mathbb{F}_q)$ -linear  $\infty$ -categories:*

$$(\mathcal{D}_Y^{(\nu)}\text{-Mod}^{\text{Fr}^*})_{\text{lfg-u},\circ}^{\text{op}} \xrightarrow{\sim} W_n(\mathbb{F}_q)\text{-Shv}_c(Y_0).$$

*Proof.* It follows from Proposition 1.12 that we have an equivalence:

$$\begin{aligned} (\mathcal{D}_Y^{(\nu)}\text{-Mod}^{\text{Fr}^*})_{\text{lfg-u},\circ} &\xrightarrow{\sim} D_{\text{lfg-u},\circ}^b(\mathcal{D}_{Y,F}^{(\nu)}\text{-Mod}^{\heartsuit}) \\ &\xrightarrow{\sim} D_{\text{lfg-u},\circ}^b(\mathcal{D}_{Y,F}\text{-Mod}^{\heartsuit}) \xrightarrow{\sim} W_n(\mathbb{F}_q)\text{-Shv}_c(Y_0)^{\text{op}}, \end{aligned}$$

where the last step is the Riemann–Hilbert correspondence (1.3).  $\square$

**Remark 1.14.** If we worked with triangulated categories instead of stable  $\infty$ -categories, the resulting fixed point category would be defined incorrectly. This is the main reason we use  $\infty$ -categories in this paper.

## 1.5. Proof of Proposition 1.12.

1.5.1. We first recall a basic adjunction for  $F$ -crystals:

$$\begin{array}{ccc} & \mathcal{D}_{F,Y}^{(\nu)} \otimes_{\mathcal{D}_Y} - & \\ & \uparrow & \\ \mathcal{D}_Y^{(\nu)}\text{-Mod}^{\heartsuit} & \xrightarrow{\quad} & \mathcal{D}_{F,Y}^{(\nu)}\text{-Mod}^{\heartsuit} \\ & \downarrow \text{obl} & \end{array} \quad (1.7)$$

Since  $\mathcal{D}_{F,Y}^{(\nu)}$  is flat as a right  $\mathcal{D}_Y^{(\nu)}$ -module (in fact, locally free), the left adjoint is exact. In particular, the right adjoint preserves injective objects. Following the terminology of [EK04, §13.6], we call a  $\mathcal{D}_{F,Y}^{(\nu)}$ -module *induced*, if it arises from applying the functor  $\mathcal{D}_{F,Y}^{(\nu)} \otimes_{\mathcal{D}_Y^{(\nu)}} -$  to a  $\mathcal{D}_Y^{(\nu)}$ -module.

1.5.2. Each  $F$ -crystal  $(\mathcal{M}, \psi_{\mathcal{M}})$  admits a canonical two-step resolution by induced  $F$ -crystals (Proposition 13.6.1 of *loc.cit.*):

$$0 \rightarrow \mathcal{D}_{F,Y}^{(\nu)} \otimes_{\mathcal{D}_Y^{(\nu)}} \text{Fr}_Y^* \mathcal{M} \xrightarrow{\alpha} \mathcal{D}_{F,Y}^{(\nu)} \otimes_{\mathcal{D}_Y^{(\nu)}} \mathcal{M} \rightarrow \mathcal{M} \rightarrow 0, \quad (1.8)$$

where  $\alpha$  is the *difference* between the following two maps:

(a) The adjoint of the map of  $\mathcal{D}_Y^{(\nu)}$ -modules:

$$\text{Fr}_Y^* \mathcal{M} \xrightarrow{\sim} \text{Fr}_Y^* \mathcal{D}_Y^{(\nu)} \otimes_{\mathcal{D}_Y^{(\nu)}} \mathcal{M} \hookrightarrow \mathcal{D}_{F,Y}^{(\nu)} \otimes_{\mathcal{D}_Y^{(\nu)}} \mathcal{M}.$$

(b) The adjoint of the map of  $\mathcal{D}_Y^{(\nu)}$ -modules:

$$\mathrm{Fr}_Y^* \mathcal{M} \xrightarrow{\psi_{\mathcal{M}}} \mathcal{M} \hookrightarrow \mathcal{D}_{F,Y}^{(\nu)} \otimes_{\mathcal{D}_Y^{(\nu)}} \mathcal{M}.$$

**Remark 1.15.** Although  $\alpha$  is a map between induced  $F$ -crystals, it is not induced from a map of  $\mathcal{D}_Y^{(\nu)}$ -modules.

1.5.3. Suppose  $\mathcal{C}$  is a stable  $\infty$ -category equipped with a right complete  $t$ -structure, and  $\mathcal{C}^\heartsuit$  has enough injective objects. Then the canonical functor  $D^+(\mathcal{C}^\heartsuit) \rightarrow \mathcal{C}^+$  is an equivalence if and only if the following conditions holds:

- For every injective object  $I \in \mathcal{C}^\heartsuit$ , the abelian group  $\mathrm{Ext}_{\mathcal{C}}^i(\mathcal{M}, I) = 0$  for all  $\mathcal{M} \in \mathcal{C}^\heartsuit$  and  $i \geq 1$ .

This is essentially (the dual of) [Lu17, Proposition 1.3.3.7].

1.5.4. We now combine the above ingredients to give a proof of Proposition 1.12. We first note that  $\mathcal{D}_Y^{(\nu)}\text{-Mod}^{\mathrm{Fr}^* \rightarrow \mathrm{id}}$  is right complete with respect to its  $t$ -structure (Lemma 1.10). Let  $I, \mathcal{M} \in \mathcal{D}_{F,Y}^{(\nu)}\text{-Mod}^\heartsuit$ , with  $I$  being an injective object. By Lemma 1.9, the Hom-complex  $R\mathrm{Hom}_{\mathcal{D}_Y^{(\nu)}\text{-Mod}^{\mathrm{Fr}^* \rightarrow \mathrm{id}}}(\mathcal{M}, I)$  identifies with the (homotopy) fiber of:

$$\beta : R\mathrm{Hom}_{\mathcal{D}_Y^{(\nu)}\text{-Mod}}(\mathcal{M}, I) \rightarrow R\mathrm{Hom}_{\mathcal{D}_Y^{(\nu)}\text{-Mod}}(\mathrm{Fr}_Y^* \mathcal{M}, I), \quad f \rightsquigarrow \psi_I \circ \mathrm{Fr}_Y^*(f) - f \circ \psi_{\mathcal{M}}.$$

We must show that this fiber has zero cohomology groups in degree  $i \geq 1$ .

1.5.5. From the discussion in §1.5.1, we know that  $I$  is also injective as an object in  $\mathcal{D}_Y^{(\nu)}\text{-Mod}$ . Hence we may replace  $\beta$  by a map between *underived* Hom-spaces:

$$\beta : \mathrm{Hom}_{\mathcal{D}_Y^{(\nu)}\text{-Mod}^\heartsuit}(\mathcal{M}, I) \rightarrow \mathrm{Hom}_{\mathcal{D}_Y^{(\nu)}\text{-Mod}^\heartsuit}(\mathrm{Fr}_Y^* \mathcal{M}, I).$$

We now observe that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}_Y^{(\nu)}\text{-Mod}^\heartsuit}(\mathcal{M}, I) & \xrightarrow{\beta} & \mathrm{Hom}_{\mathcal{D}_Y^{(\nu)}\text{-Mod}^\heartsuit}(\mathrm{Fr}_Y^* \mathcal{M}, I) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Hom}_{\mathcal{D}_{F,Y}^{(\nu)}\text{-Mod}^\heartsuit}(\mathcal{D}_{F,Y}^{(\nu)} \otimes_{\mathcal{D}_Y^{(\nu)}} \mathrm{Fr}_Y^* \mathcal{M}, I) & \xrightarrow{\alpha^*} & \mathrm{Hom}_{\mathcal{D}_{F,Y}^{(\nu)}\text{-Mod}^\heartsuit}(\mathcal{D}_{F,Y}^{(\nu)} \otimes_{\mathcal{D}_Y^{(\nu)}} \mathcal{M}, I) \end{array}$$

Here, the vertical isomorphisms come from the adjunction (1.7). The bottom map is precomposition with  $\alpha$  in the two-step resolution (1.8). This calculation shows that the fiber of  $\beta$  is isomorphic to the abelian group  $\mathrm{Hom}_{\mathcal{D}_{F,Y}^{(\nu)}\text{-Mod}^\heartsuit}(\mathcal{M}, I)$ , as expected.  $\square$ (Proposition 1.12)

## 2. THE $p$ -TORSION ARTIN RECIPROCITY FUNCTOR

In this section, we construct  $p$ -torsion Artin reciprocity as a contravariant functor from  $\mathrm{Coh}(\widetilde{\mathrm{Jac}}_n^{\mathrm{id}})_{\mathcal{O}}^{\mathrm{id}}$  to the category of  $W_n(\mathbb{F}_q)$ -sheaves on the  $\mathrm{Jac}_n$ . Then we verify that this functor categorifies the usual Artin reciprocity map for  $p$ -torsion characters trivial on  $x_0$ . In fact, our construction applies more generally to any abelian scheme  $A$  over  $W_n(\mathbb{F}_q)$  and the Verschiebung-fixed locus of the universal additive extension of  $A^\vee$ .

Along the way, we will discuss how the Verschiebung-fixed locus  $\widetilde{\mathrm{Jac}}_n^{\mathrm{id}}$  is related to characters of the étale fundamental group.

### 2.1. The Jacobian scheme.

2.1.1. From now on, we will assume  $k = \mathbb{F}_q$  for simplicity. The ring of length- $n$  Witt vectors will be denoted by  $W_n$ . Let  $X$  be a smooth, proper, geometrically connected curve over  $\mathbb{F}_q$ . We assume that it contains an  $\mathbb{F}_q$ -rational point  $x_0$ . We fix a smooth lift  $X_n$  over  $W_n$ ; it always exists since the obstruction to such lifts lies in a degree-2 cohomology group. The smoothness implies that  $x_0$  can be lifted to a  $W_n$ -rational point of  $X_n$ .

2.1.2. The Jacobian  $\text{Jac}_n$  parametrizes degree-0 line bundles on  $X_n$  together with a rigidification at  $x_0$ . It is representable by a smooth abelian scheme over  $W_n$  (see, for example, [Kl05, Proposition 9.5.19]). There is a canonical self-duality of  $\text{Jac}_n$ , as witnessed by the following Poincaré line bundle  $\mathcal{P}$  on  $\text{Jac}_n \times_{W_n} \text{Jac}_n$ :

$$\mathcal{L}_1, \mathcal{L}_2 \rightsquigarrow \det R\Gamma(X_n, \mathcal{L}_1 \otimes \mathcal{L}_2) \otimes \det R\Gamma(X_n, \mathcal{L}_1)^{\otimes -1} \otimes \det R\Gamma(X_n, \mathcal{L}_2)^{\otimes -1} \otimes \det R\Gamma(X_n, \mathcal{O}).$$

The notation  $R\Gamma(X_n, -)$  is a short-hand for pushforward along  $X_n \times_{W_n} S \rightarrow S$ , where  $S$  is a test  $W_n$ -scheme. The determinant is well-defined since  $R\Gamma(X_n, -)$ , applied to any coherent sheaf flat over  $S$ , yields a perfect complex.

2.1.3. To each abelian scheme  $A$  over  $W_n$ , we attach an additive extension of its dual:

$$0 \rightarrow H^0(\Omega_{A/W_n}) \rightarrow \tilde{A}^\vee \xrightarrow{\pi} A^\vee \rightarrow 0.$$

Explicitly,  $\tilde{A}^\vee$  classifies a multiplicative line bundle on  $A$  equipped with an integrable connection. In other words, an  $S$ -point of  $\tilde{A}^\vee$  is a multiplicative line bundle on  $S \times_{W_n} A$  whose  $\mathcal{O}$ -module structure enhances to that of  $\mathcal{D}_{S \times A/S}^{(0)}$ . We note that this datum is equivalent to a multiplicative pair  $(\mathcal{L}, \nabla)$ , where  $\mathcal{L}$  is a line bundle on  $S \times_{W_n} A$  and  $\nabla$  is an integrable connection on  $\mathcal{L}$  along  $A$  ([La96, Lemme 2.1.1]). One knows that  $\tilde{A}^\vee$  is representable by a smooth scheme of relative dimension  $2g$  over  $W_n$ , and is in fact the universal additive extension of  $A^\vee$  [MM06, §1]. Pulling back along the Frobenius on  $A$  (see §1.1.5) defines an endomorphism:

$$\text{Ver} : \tilde{A}^\vee \rightarrow \tilde{A}^\vee, \tag{2.1}$$

which we will call the *Verschiebung* endomorphism.

**Remark 2.1.** Suppose  $n = 1$ . Then (2.1) lifts the usual *Verschiebung* endomorphism of the abelian variety  $A^\vee$  over  $\mathbb{F}_q$ .

2.1.4. We let  $\tilde{A}^{\vee, \natural}$  denote the *Verschiebung*-fixed point locus of  $\tilde{A}^\vee$ . In other words, it is the *a priori* derived closed subscheme given by the fiber product:

$$\begin{array}{ccc} \tilde{A}^{\vee, \natural} & \xrightarrow{\iota} & \tilde{A}^\vee \\ \downarrow & & \downarrow \Delta \\ \tilde{A}^\vee & \xrightarrow{(\text{Ver}, \text{id})} & \tilde{A}^\vee \times_{W_n} \tilde{A}^\vee \end{array} \tag{2.2}$$

**Lemma 2.2.** *The map  $\iota : \tilde{A}^{\vee, \natural} \hookrightarrow \tilde{A}^\vee$  is a regular immersion of classical schemes.*

*Proof.* By smoothness of  $\tilde{A}^\vee$  over  $W_n$ , we see that both embeddings of  $\tilde{A}^\vee$  in  $\tilde{A}^\vee \times_{W_n} \tilde{A}^\vee$  in (2.2) are regular immersions of half-dimensional subschemes; indeed, these are both sections of the projection map  $\text{pr}_2 : \tilde{A}^\vee \times_{W_n} \tilde{A}^\vee \rightarrow \tilde{A}^\vee$  which is smooth (c.f. [Stacks, 067R]). Recall that

given a Noetherian local ring  $(R, \mathfrak{m})$  of dimension  $d$  and elements  $f_1, \dots, f_d \in \mathfrak{m}$ , the following statements are equivalent:

- (a) the Koszul complex associated to  $f_1, \dots, f_d$  has cohomology only in degree 0 (i.e.,  $f_1, \dots, f_d$  is a regular sequence);
- (b) the Krull dimension of  $R/(f_1, \dots, f_d)$  vanishes.

Using this fact, we see that the Lemma will follow from checking the Krull dimension of (the underlying classical scheme) of  $\widetilde{A}^{\vee, \natural}$  vanishes. This statement can in turn be verified after base change to the special fiber over  $\text{Spec}(\mathbb{F}_q)$ , so we may assume  $n = 1$ . In this case, we have the usual Verschiebung endomorphism on  $A^\vee$ , given by the Frobenius-pullback of line bundles on  $A$ . There is a commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(\Omega_{A/\mathbb{F}_q}) & \longrightarrow & \widetilde{A}^\vee & \xrightarrow{\pi} & A^\vee \longrightarrow 0 \\ & & \downarrow \text{Fr}^* = 0 & & \downarrow \text{Ver} & & \downarrow \text{Ver} \\ 0 & \longrightarrow & H^0(\Omega_{A/\mathbb{F}_q}) & \longrightarrow & \widetilde{A}^\vee & \xrightarrow{\pi} & A^\vee \longrightarrow 0 \end{array}$$

Consequently,  $\pi$  identifies  $\widetilde{A}^{\vee, \natural}$  with the Verschiebung-fixed point scheme  $A^{\vee, \natural}$  of  $A^\vee$ . We now show that the latter is zero-dimensional. Note that  $A^{\vee, \natural}$  is the fiber of the map of abelian varieties:

$$0 \rightarrow A^{\vee, \natural} \rightarrow A^\vee \xrightarrow{\text{Ver} - \text{id}} A^\vee.$$

Now, the endomorphism  $\text{Ver} - \text{id}$  is an isogeny since it is dual to the Lang isogeny  $\text{Fr} - \text{id} : A \rightarrow A$ , so we are done.  $\square$

**Remark 2.3.** In particular, the proof shows that  $\widetilde{A}^{\vee, \natural}$  is a finite  $W_n$ -scheme whose  $\mathbb{F}_q$ -points are identified with those of the Verschiebung fixed locus of the special fiber  $A_1^\vee$  of  $A^\vee$ .

2.1.5. Let  $\widetilde{\text{Jac}}_n$  denote the abelian scheme classifying degree-0, rigidified line bundles on  $X_n$  together with an integrable connection (relative to  $W_n$ ). Then the auto-duality of  $\text{Jac}_n$  enhances to an isomorphism of  $\widetilde{\text{Jac}}_n$  with the universal additive extension of the dual of  $\text{Jac}_n$ . In other words, we have a commutative diagram:

$$\begin{array}{ccc} \widetilde{\text{Jac}}_n & \xrightarrow{\sim} & \widetilde{\text{Jac}}_n^\vee \\ \downarrow & & \downarrow \\ \text{Jac}_n & \xrightarrow{\sim} & \text{Jac}_n^\vee \end{array}$$

Furthermore, the Verschiebung endomorphism on  $\widetilde{\text{Jac}}_n^\vee$  passes to the endomorphism:

$$\text{Ver} : \widetilde{\text{Jac}}_n \rightarrow \widetilde{\text{Jac}}_n, \quad \mathcal{L} \rightsquigarrow \text{Fr}_{X_n}^* \mathcal{L}.$$

2.1.6. We write  $\widetilde{\text{Jac}}_n^\natural$  for the Verschiebung-fixed point locus of  $\widetilde{\text{Jac}}_n$ . Explicitly, an  $S$ -point of  $\widetilde{\text{Jac}}_n^\natural$  is a degree-0, rigidified line bundle  $\mathcal{L}$  on  $X_n$  with an integrable connection, as well as an identification  $\text{Fr}_{X_n}^* \mathcal{L} \xrightarrow{\sim} \mathcal{L}$  of  $\mathcal{D}_{S \times X_n/S}^{(0)}$ -modules compatible with the rigidification at  $x_0$ . Lemma 2.2 shows that  $\widetilde{\text{Jac}}_n^\natural$  is a disjoint union of connected zero-dimensional schemes  $\widetilde{\text{Jac}}_{n, \sigma}^\natural$ , where  $\sigma$  ranges through the  $\mathbb{F}_q$ -points of  $\widetilde{\text{Jac}}_n^\natural$ .

## 2.2. Galois deformations.

2.2.1. In this subsection, we relate the Verschiebung-fixed point locus to Galois deformations over  $W_n$ . This relation is not needed for the construction of the Artin reciprocity functor.

2.2.2. Fix a geometric point  $\bar{x}_0$  of the curve  $X$  lying over  $x_0$ . We use  $\bar{x}_0$  to define the étale fundamental groups of  $\bar{X}$  and  $X$ . There is a short exact sequence:

$$1 \rightarrow \pi_1(\bar{X}) \rightarrow \pi_1(X) \rightarrow \pi_1(\mathrm{Spec}(\mathbb{F}_q)) \rightarrow 0,$$

where the second map has a section defined by  $x_0$ . We will be concerned with continuous characters  $\sigma : \pi_1(X) \rightarrow \mathbb{F}_q^\times$  which are *trivial on*  $\pi_1(x_0)$ . We note that such objects are equivalently rank-1 étale  $\mathbb{F}_q$ -sheaves on  $X$  trivialized at  $x_0$ . By Katz's equivalence, they are in turn  $\mathbb{F}_q$ -points of  $\widetilde{\mathrm{Jac}}_1^{\natural}$  (or equivalently  $\widetilde{\mathrm{Jac}}_n^{\natural}$ ).

**Remark 2.4.** We know *a fortiori* that  $\sigma$  is equivalent to a character of the geometric fundamental group  $\pi_1(\bar{X})$ , but viewing it as such is less natural for our purpose.

2.2.3. Let  $\sigma : \pi_1(X) \rightarrow \mathbb{F}_q^\times$  be a continuous character which is trivial on  $\pi_1(x_0)$ . We define a functor  $\mathrm{Def}_n^\sigma$  on the category of Artinian, local  $W_n$ -algebras with residue field  $\mathbb{F}_q$  as follows. For  $S = \mathrm{Spec}(A)$  where  $A$  is such a  $W_n$ -algebra, we let  $\mathrm{Def}_n^\sigma(A)$  be the set of continuous characters:

$$\pi_1(X) \rightarrow A^\times, \quad \text{trivial on } \pi_1(x_0),$$

such that its reduction along  $A^\times \rightarrow \mathbb{F}_q^\times$  identifies with  $\sigma$ . Since  $A$  is finite, the continuity requirement is equivalent to factoring through a finite quotient.

2.2.4. We now use regard  $\sigma$  as an  $\mathbb{F}_q$ -point of  $\widetilde{\mathrm{Jac}}_n^{\natural}$ .

**Lemma 2.5.** *The functor  $\mathrm{Def}_n^\sigma$  is represented by the pointed  $W_n$ -scheme  $\widetilde{\mathrm{Jac}}_{n,\sigma}^{\natural}$ .*

*Proof.* Given a Artinian, local  $W_n$ -algebra  $A$  with residue field  $\mathbb{F}_q$ , write  $S = \mathrm{Spec}(A)$ . An element of  $\mathrm{Def}_n^\sigma(A)$  is equivalent to a rigidified étale  $A$ -local system on  $X_n$  of rank 1, whose induced  $\mathbb{F}_q$ -local system identifies with  $\sigma$ . By Lemma 1.4, this datum is equivalent to a line bundle  $\mathcal{L}$  over  $S \times_{W_n} X_n$  together with a connection along  $X_n$  and an isomorphism  $\mathcal{L} \xrightarrow{\sim} (\mathrm{id}_S \times \mathrm{Fr}_X)^* \mathcal{L}$ , whose restriction to  $x_0$  is rigidified and whose reduction to  $\mathbb{F}_q$  identifies with  $\sigma$ . The latter is precisely that of an  $S$ -point of  $\widetilde{\mathrm{Jac}}_n^{\natural}$  whose closed point is  $\sigma$ .  $\square$

2.2.5. In particular, we obtain a fully faithful functor:

$$\mathrm{QCoh}(\mathrm{Def}_n^\sigma) \xrightarrow{\sim} \mathrm{QCoh}(\widetilde{\mathrm{Jac}}_{n,\sigma}^{\natural}) \hookrightarrow \mathrm{QCoh}(\widetilde{\mathrm{Jac}}_n^{\natural}) \quad (2.3)$$

**Remark 2.6.** The argument of Lemma 2.5 works more generally for representations of higher rank. Namely, for a rank- $r$  representation  $\sigma$ , the functor of Galois deformations  $\mathrm{Def}_n^\sigma$  over  $W_n$  is represented by the formal neighborhood of the  $W_n$ -stack  $\mathrm{LocSys}_{r,n}^{\natural}$  at  $\sigma$ , regarded as a pointed  $W_n$ -stack. However, for  $r \geq 2$ ,  $\mathrm{LocSys}_{r,n}^{\natural}$  may no longer be zero-dimensional, so it contains more information than  $\mathrm{Def}_n^\sigma$  (c.f. Y. Laszlo [La01]).

### 2.3. Fourier–Mukai–Laumon transform.

2.3.1. Let  $A$  be an abelian scheme over  $W_n$ . We recall that the Fourier–Mukai transform defines an equivalence of  $W_n$ -linear  $\infty$ -categories:

$$\Phi : \mathrm{QCoh}(A^\vee) \xrightarrow{\sim} \mathrm{QCoh}(A), \quad \mathcal{F} \rightsquigarrow Rq_*(p^* \mathcal{F} \otimes^L \mathcal{P}), \quad (2.4)$$

where  $\mathcal{P}$  is the universal line bundle on  $A^\vee \times_{W_n} A$  and  $p, q$  are the two projections.

2.3.2. The Fourier–Mukai–Laumon transform (c.f. [La96, Théorème 3.2.1]) enhances (2.4) into an equivalence between  $\mathrm{QCoh}(\tilde{A}^\vee)$  and  $\mathcal{D}^{(0)\text{-Mod}}(A)$ . More precisely, we write  $\tilde{\mathcal{P}}$  for the universal line bundle on  $\tilde{A}^\vee \times_{W_n} A$  equipped with an integrable connection along  $A$  and  $\tilde{p}, \tilde{q}$  for the two projections.

**Lemma 2.7.** *The functor:*

$$\tilde{\Phi} : \mathrm{QCoh}(\tilde{A}^\vee) \rightarrow \mathcal{D}^{(0)\text{-Mod}}(A), \quad \mathcal{F} \rightsquigarrow R\tilde{q}_*(\tilde{p}^*\mathcal{F} \otimes^L \tilde{\mathcal{P}}) \quad (2.5)$$

defines an equivalence of  $W_n$ -linear  $\infty$ -categories, making the following diagram commutes:

$$\begin{array}{ccc} \mathrm{QCoh}(\tilde{A}^\vee) & \xrightarrow{\tilde{\Phi}} & \mathcal{D}^{(0)\text{-Mod}}(A) \\ \downarrow \pi_* & & \downarrow \mathrm{oblv}_{\mathcal{D}^{(0)}} \\ \mathrm{QCoh}(A^\vee) & \xrightarrow{\Phi} & \mathrm{QCoh}(A) \end{array} \quad (2.6)$$

The result in *loc.cit.* is proved when the base is in characteristic zero, although this assumption is superfluous. We indicate the necessary modification of Laumon’s argument which proves the result for an abelian scheme  $A$  over any locally Noetherian base scheme  $S$ . (We then apply the result to  $S = W_n$ .)

*Sketch of proof.* The Lemma follows formally from two canonical isomorphisms:

(a) The  $\mathcal{D}^{(0)}$ -module pushforward along  $\tilde{p}$  of  $\tilde{\mathcal{P}}$ :

$$\tilde{p}_{*,\mathrm{dR}}(\tilde{\mathcal{P}}) \xrightarrow{\sim} R\tilde{p}_*(\tilde{\mathcal{P}} \xrightarrow{\nabla} \tilde{\mathcal{P}} \otimes \Omega_A^1 \xrightarrow{\nabla} \cdots)$$

ought to be identified with  $\tilde{\epsilon}_* \mathcal{O}_S[-g]$ , where  $\tilde{\epsilon} : S \rightarrow \tilde{A}^\vee$  is the identity;

(b) The pushforward along  $\tilde{q}$  of  $\tilde{\mathcal{P}}$  ought to be identified with  $\epsilon_{*,\mathrm{dR}}(\mathcal{O}_S)$ , where  $\epsilon_{*,\mathrm{dR}}$  is the  $\mathcal{D}^{(0)}$ -module pushforward along the identity  $\epsilon : S \rightarrow A$ .<sup>6</sup>

The argument in *loc.cit.* works verbatim for the first identification, but it appeals to Kashiwara’s lemma for the second identification, which does *not* hold outside characteristic zero. We will provide this isomorphism in a more direct manner. First, there is a canonically defined morphism of  $\mathcal{D}_{A/S}^{(0)}$ -modules:

$$\epsilon_{*,\mathrm{dR}}(\mathcal{O}_S) \rightarrow R\tilde{q}_*\tilde{\mathcal{P}}, \quad (2.7)$$

obtained from adjunction by a map of  $\mathcal{O}_S$ -modules  $\mathcal{O}_S \rightarrow L\epsilon^*R\tilde{q}_*\tilde{\mathcal{P}}$ , which in turn comes from base change. Next, we observe that  $R\tilde{q}_*\tilde{\mathcal{P}}$  identifies with  $Rq_*(\mathcal{O}_{\tilde{A}^\vee} \otimes \mathcal{P})$ . By §2.3 of *loc.cit.*,  $\mathcal{O}_{\tilde{A}^\vee}$  admits a filtration  $F^{\leq i} \mathcal{O}_{\tilde{A}^\vee}$  as an  $\mathcal{O}_{A^\vee}$ -module, such that:

$$F^{\leq i} \mathcal{O}_{\tilde{A}^\vee} / F^{\leq i-1} \mathcal{O}_{\tilde{A}^\vee} \xrightarrow{\sim} \mathrm{Sym}^i(\epsilon^* \mathcal{J}_{A/S}) \otimes_{\mathcal{O}_S} \mathcal{O}_{A^\vee}.$$

Since  $Rq_*(\mathcal{P})$  is canonically isomorphic to  $\epsilon_* \epsilon^* \omega_{A/S}^{\otimes -1}[-g]$  (Lemme 1.2.5 of *loc.cit.*), we see that  $R\tilde{q}_*\tilde{\mathcal{P}}$  is concentrated in cohomological degree  $g$  and inherits a filtration by  $\mathcal{O}_S$ -submodules. On the other hand,  $\epsilon_{*,\mathrm{dR}}(\mathcal{O}_S)$  admits a filtration as an  $\mathcal{O}_S$ -module by order of differential operators in  $\mathcal{D}_{A/S}^{(0)}$ , whose associated graded pieces are given by:

$$F^{\leq i} \epsilon_{*,\mathrm{dR}}(\mathcal{O}_S) / F^{\leq i-1} \epsilon_{*,\mathrm{dR}}(\mathcal{O}_S) \xrightarrow{\sim} \mathrm{Sym}^i(\epsilon^* \mathcal{J}_{A/S}) \otimes_{\mathcal{O}_S} \epsilon_* \epsilon^* \omega_{A/S}^{\otimes -1}[-g];$$

<sup>6</sup>We remark that  $\epsilon_{*,\mathrm{dR}}(\mathcal{O}_S)$ , considered as a left  $\mathcal{D}^{(0)}$ -module, lives in cohomological degree  $g$ .

here, the factor  $\omega_{A/S}^{\otimes -1}[-g]$  arises from passing the  $\mathcal{D}_{A/S}^{(0)}$ -action from right to left. The morphism (2.7) is compatible with the filtrations, i.e., the following diagram commutes:

$$\begin{array}{ccc} F^{\leq i} \epsilon_{*, \text{dR}}(\mathcal{O}_S) & \longrightarrow & F^{\leq i} R\tilde{q}_* \tilde{\mathcal{P}} \\ \downarrow & & \downarrow \\ \epsilon_{*, \text{dR}}(\mathcal{O}_S) & \longrightarrow & R\tilde{q}_* \tilde{\mathcal{P}} \end{array}$$

Indeed, this is a consequence of the two facts below:

(a) the adjoint morphism  $\mathcal{O}_S \rightarrow L\epsilon^* Rq_*(\mathcal{O}_{\tilde{A}^\vee} \otimes \mathcal{P})$  of (2.7) factors through

$$L\epsilon^* Rq_*(F^{\leq 0} \mathcal{O}_{\tilde{A}^\vee} \otimes \mathcal{P}) \cong L\epsilon^* Rq_* \mathcal{P};$$

(b) the filtration on  $R\tilde{q}_* \tilde{\mathcal{P}}$  is compatible with  $\mathcal{D}_{A/S}^{(0)}$ -action, in the sense that:

$$F^{\leq i} \mathcal{D}_{A/S}^{(0)} \cdot F^{\leq j} R\tilde{q}_* \tilde{\mathcal{P}} \subset F^{\leq i+j} R\tilde{q}_* \tilde{\mathcal{P}}.$$

Thus it remains to show that (2.7) induces an isomorphism on each graded piece. On the  $i$ th associated graded piece, it induces the following map:

$$\text{Sym}^i(\epsilon^* \mathcal{T}_{A/S}) \otimes_{\mathcal{O}_S} \epsilon_* \epsilon^*(\omega_{A/S}^{\otimes -1})[-g] \xrightarrow{\text{id} \otimes \alpha} \text{Sym}^i(\epsilon^* \mathcal{T}_{A/S}) \otimes_{\mathcal{O}_S} Rq_* \mathcal{P},$$

where  $\alpha$  is the canonical isomorphism  $\epsilon_* \epsilon^*(\omega_{A/S}^{\otimes -1})[-g] \xrightarrow{\sim} Rq_* \mathcal{P}$ .  $\square$

2.3.3. We show that the Fourier–Mukai–Laumon transform satisfies the Frobenius compatibility alluded to in the introduction. We emphasize that there is no analogous diagram for the usual Fourier–Mukai transform  $\Phi$  beyond the special fiber.

**Lemma 2.8.** *The following diagram commutes:*

$$\begin{array}{ccc} \text{QCoh}(\tilde{A}^\vee) & \xrightarrow{\tilde{\Phi}} & \mathcal{D}^{(0)\text{-Mod}}(A) \\ \downarrow \text{Ver}_* & & \downarrow \text{Fr}_A^* \\ \text{QCoh}(\tilde{A}^\vee) & \xrightarrow{\tilde{\Phi}} & \mathcal{D}^{(0)\text{-Mod}}(A) \end{array} \quad (2.8)$$

*Proof.* By definition of the Verschiebung endomorphism, we have an isomorphism:

$$(\text{Ver} \times \text{id})^* \tilde{\mathcal{P}} \xrightarrow{\sim} \text{Fr}_A^* \tilde{\mathcal{P}} \quad (2.9)$$

of  $\mathcal{D}_{\tilde{A}^\vee \times A / \tilde{A}^\vee}^{(0)}$ -modules. Now, consider the following commutative diagram:

$$\begin{array}{ccccc} & & \tilde{A}^\vee \times A & & \\ & \swarrow \tilde{p} & \uparrow \text{Ver} \times \text{id} & \searrow \tilde{q} & \\ \tilde{A}^\vee & & \tilde{A}^\vee \times A & & A \\ \uparrow \text{Ver} & \swarrow \tilde{p} & \uparrow \tilde{p} & \searrow \tilde{q} & \downarrow \text{Fr}_A \\ \tilde{A}^\vee & & \tilde{A}^\vee \times A & & A \\ & \swarrow \tilde{p} & \downarrow \text{id} \times \text{Fr}_A & \searrow \tilde{q} & \\ & & \tilde{A}^\vee \times A & & A \end{array}$$

where the dotted arrows indicate that  $\text{Fr}_A$  is only well defined affine locally on  $A$ , but suffices to define the pullback of  $\mathcal{D}^{(0)}$ -modules. Using the projection formula, both circuits in (2.8) can

be identified with pulling back along  $\tilde{p}$ , tensoring with the  $\mathcal{D}_{\tilde{A}^\vee \times A/\tilde{A}^\vee}^{(0)}$ -module (2.9), and then pushing forward along  $\tilde{q}$ .  $\square$

2.3.4. We now let  $\mathrm{QCoh}(\tilde{A}^\vee)^{\mathrm{Ver}_*}$  and  $\mathcal{D}^{(0)\text{-Mod}}(A)^{\mathrm{Fr}_*}$  be the categorical fixed points (c.f. §1.3) of these  $W_n$ -linear  $\infty$ -categories under the endomorphisms  $\mathrm{Ver}_*$  and  $\mathrm{Fr}_A^*$ . It follows from the Lemma that we have a  $W_n$ -linear equivalence:

$$\mathrm{QCoh}(\tilde{A}^\vee)^{\mathrm{Ver}_*} \xrightarrow{\sim} \mathcal{D}^{(0)\text{-Mod}}(A)^{\mathrm{Fr}_*}.$$

2.3.5. Finally, setting  $A := \mathrm{Jac}_n$  and using the auto-duality explained in §2.1, we obtain an equivalence:

$$\mathrm{QCoh}(\widetilde{\mathrm{Jac}}_n)^{\mathrm{Ver}_*} \xrightarrow{\sim} \mathcal{D}^{(0)\text{-Mod}}(\mathrm{Jac}_n)^{\mathrm{Fr}_*} \quad (2.10)$$

which makes the following diagram commute:

$$\begin{array}{ccc} \mathrm{QCoh}(\widetilde{\mathrm{Jac}}_n)^{\mathrm{Ver}_*} & \xrightarrow{(2.10)} & \mathcal{D}^{(0)\text{-Mod}}(\mathrm{Jac}_n)^{\mathrm{Fr}_*} \\ \downarrow \mathrm{oblv} & & \downarrow \mathrm{oblv} \\ \mathrm{QCoh}(\widetilde{\mathrm{Jac}}_n) & \xrightarrow{\tilde{\Phi}} & \mathcal{D}^{(0)\text{-Mod}}(\mathrm{Jac}_n) \end{array}$$

## 2.4. The $p$ -torsion Artin reciprocity functor.

2.4.1. Consider the  $\infty$ -category consisting of an object  $\mathcal{F} \in \mathrm{Coh}(\widetilde{\mathrm{Jac}}_n^{\natural})$  together with an auto-morphism  $\alpha$ . It can be regarded as fixed points under the identity endomorphism on  $\mathrm{Coh}(\widetilde{\mathrm{Jac}}_n^{\natural})$ , denoted by  $\mathrm{Coh}(\widetilde{\mathrm{Jac}}_n^{\natural})^{\mathrm{id}}$ . Note that there is a functor:

$$\mathrm{Coh}(\widetilde{\mathrm{Jac}}_n^{\natural})^{\mathrm{id}} \rightarrow \mathrm{Coh}(\widetilde{\mathrm{Jac}}_n)^{\mathrm{Ver}_*}, \quad (2.11)$$

defined by sending  $(\mathcal{F}, \alpha)$  to the coherent sheaf  $\iota_*\mathcal{F}$  together with the isomorphism:

$$\iota_*\mathcal{F} \xrightarrow{\iota_*\alpha} \iota_*\mathcal{F} \xrightarrow{\sim} \mathrm{Ver}_* \iota_*\mathcal{F}.$$

Here,  $\iota$  denotes the closed immersion of  $\widetilde{\mathrm{Jac}}_n^{\natural}$  in  $\widetilde{\mathrm{Jac}}_n$ .

2.4.2. To summarize, we have constructed a commutative diagram:

$$\begin{array}{ccc} \mathrm{Coh}(\widetilde{\mathrm{Jac}}_n^{\natural})^{\mathrm{id}} & \xrightarrow{(2.11)} & \mathrm{Coh}(\widetilde{\mathrm{Jac}}_n)^{\mathrm{Ver}_*} \xrightarrow{(2.10)} \mathcal{D}^{(0)\text{-Mod}}(\mathrm{Jac}_n)^{\mathrm{Fr}_*} \\ \downarrow \mathrm{oblv} & & \downarrow \mathrm{oblv} \\ \mathrm{Coh}(\widetilde{\mathrm{Jac}}_n) & \xrightarrow{\tilde{\Phi}} & \mathcal{D}^{(0)\text{-Mod}}(\mathrm{Jac}_n) \\ \downarrow \pi_* & & \downarrow \mathrm{oblv}_{\mathcal{D}^{(0)}} \\ \mathrm{Coh}(\mathrm{Jac}_n) & \xrightarrow{\Phi} & \mathrm{QCoh}(\mathrm{Jac}_n) \end{array} \quad (2.12)$$

2.4.3. We write  $\mathrm{Coh}(\widetilde{\mathrm{Jac}}_n^{\natural})_{\circ}^{\mathrm{id}}$  for the full subcategory of  $\mathrm{Coh}(\widetilde{\mathrm{Jac}}_n^{\natural})^{\mathrm{id}}$  consisting of  $(\mathcal{F}, \alpha)$  where  $\mathcal{F}$  has finite Tor dimension over the base ring  $W_n$ .

**Lemma 2.9.** *Under the upper horizontal arrow of (2.12), the essential image of  $\mathrm{Coh}(\widetilde{\mathrm{Jac}}_n^{\natural})_{\circ}^{\mathrm{id}}$  lies inside  $(\mathcal{D}^{(0)\text{-Mod}}(\mathrm{Jac}_n)^{\mathrm{Fr}_*})_{\mathrm{lf-g-u}, \circ}$ .*

We refer the reader to §1.4 for the notation.

*Proof.* Take  $(\mathcal{F}, \alpha) \in \mathrm{Coh}(\widetilde{\mathrm{Jac}}_n^{\natural})_{\circ}^{\mathrm{id}}$  whose image in  $\mathcal{D}^{(0)\text{-Mod}}(\mathrm{Jac}_n)^{\mathrm{Fr}_*}$  is denoted by  $\mathcal{M}$ . The underlying object in  $\mathrm{QCoh}(\mathrm{Jac}_n)$  of  $\mathcal{M}$  is *coherent* by the commutativity of the diagram (2.12). This shows that it lies inside  $(\mathcal{D}^{(0)\text{-Mod}}(\mathrm{Jac}_n)^{\mathrm{Fr}_*})_{\mathrm{lf-g-u}}$ . It remains to check that  $\mathcal{M}$  has finite Tor dimension over the structure sheaf of  $\mathrm{Jac}_n$ . We note:

- (a)  $\mathcal{M}$  has finite Tor dimension over  $W_n$ ; indeed, this is because all functors involved are  $W_n$ -linear and  $\mathcal{F}$  has finite Tor dimension over  $W_n$ . In particular, this implies that the (derived) restriction  $\mathcal{M}|_{\text{Jac}_1}$  to the special fiber is a bounded complex of coherent sheaves.
- (b) The restriction  $\mathcal{M}|_{\text{Jac}_1}$  is in fact perfect; indeed, this is because each of its cohomology groups is an  $\mathcal{O}$ -coherent  $F$ -crystal, and we may apply [EK04, Proposition 6.9.3] to conclude that it is locally free.

We let  $\mathcal{P}^\bullet \in \text{Coh}(\text{Jac}_1)^{\geq a, \leq b}$  be the perfect complex representing  $\mathcal{M}|_{\text{Jac}_1}$ . For every closed point  $y \in \text{Jac}_n$  (necessarily factoring through  $\text{Jac}_1$ ), we have

$$Li_y^* \mathcal{M} \xrightarrow{\sim} \mathcal{P}^\bullet \otimes k_y \in \text{Vect}^{\geq a, \leq b}.$$

This implies that  $\mathcal{M}$  has Tor amplitude  $[a, b]$  over  $\text{Jac}_n$  by the following general fact. □

**Lemma 2.10.** *Suppose  $Y$  is a Noetherian scheme, and  $\mathcal{M} \in \text{Coh}(Y)$  has the property that  $Li_y^* \mathcal{M} \in \text{Vect}^{\geq a, \leq b}$  for all closed points  $y \in Y$ . Then  $\mathcal{M}$  has Tor amplitude  $[a, b]$ .*

*Proof.* We immediately reduce to the case where  $Y$  is the spectrum of a local ring. By Nakayama Lemma, we may choose a free resolution  $\mathcal{P}^\bullet$  of  $\mathcal{M}$  where each differential  $\delta$  reduces to zero modulo  $\mathfrak{m}_y$ . Thus  $Li_y^* \mathcal{M} \xrightarrow{\sim} \bigoplus_i \mathcal{P}^i \otimes_{\mathcal{O}_Y} k_y[-i]$ . The hypothesis then implies that  $\mathcal{P}^i = 0$  for  $i \notin [a, b]$ . □

2.4.4. We may thus use the composition of (2.10) and (2.11) to define the  $p$ -torsion Artin reciprocity functor:

$$\begin{aligned} \mathbb{L}_n : \text{Coh}(\widetilde{\text{Jac}}_n^\natural)_{\circ}^{\text{id}} &\rightarrow (\mathcal{D}^{(0)}\text{-Mod}(\text{Jac}_n)^{\text{Fr}^*})_{\text{Ifg-u}, \circ} \\ &\xrightarrow{\sim} W_n(\mathbb{F}_q)\text{-Shv}_c(\text{Jac})^{\text{op}}, \end{aligned}$$

where the second functor is the Riemann–Hilbert correspondence of Corollary 1.13.

**Remark 2.11.** For a fixed character  $\sigma : \pi_1(X) \rightarrow \mathbb{F}_q^\times$  trivial on  $\pi_1(x_0)$ , one may also regard the  $p$ -torsion Artin reciprocity as a functor out of  $\text{Coh}(\text{Def}_n^\sigma)_{\circ}^{\text{id}}$  by precomposing  $\mathbb{L}_n$  with (2.3):

$$\mathbb{L}_n^\sigma : \text{Coh}(\text{Def}_n^\sigma)_{\circ}^{\text{id}} \hookrightarrow \text{Coh}(\widetilde{\text{Jac}}_n^\natural)_{\circ}^{\text{id}} \xrightarrow{\mathbb{L}_n} W_n(\mathbb{F}_q)\text{-Shv}_c(\text{Jac})^{\text{op}}.$$

2.4.5. Suppose now that  $X$  is equipped with a lift to the formal scheme  $W := \text{colim}_n W_n$ . By construction, the following diagram commutes:

$$\begin{array}{ccc} \text{Coh}(\widetilde{\text{Jac}}_n^\natural)_{\circ}^{\text{id}} & \xrightarrow{\mathbb{L}_n} & W_n\text{-Shv}_c(\text{Jac})^{\text{op}} \\ \uparrow \text{Res} & & \uparrow (-) \otimes_{W_{n+1}} W_n \\ \text{Coh}(\widetilde{\text{Jac}}_{n+1}^\natural)_{\circ}^{\text{id}} & \xrightarrow{\mathbb{L}_{n+1}} & W_{n+1}\text{-Shv}_c(\text{Jac})^{\text{op}} \end{array}$$

Therefore, letting  $\widetilde{\text{Jac}}_\infty^\natural$  be the formal scheme  $\text{colim}_n \widetilde{\text{Jac}}_n^\natural$ , we obtain a functor:

$$\mathbb{L}_\infty : \text{Coh}(\widetilde{\text{Jac}}_\infty^\natural)_{\circ}^{\text{id}} \rightarrow \lim_n W_n\text{-Shv}_c(\text{Jac})^{\text{op}}$$

to the derived  $\infty$ -category of  $p$ -adic constructible sheaves.

## 2.5. Hecke eigen-property.

2.5.1. We recall that the Artin reciprocity map in geometric class field theory is a homomorphism of pro-finite groups:  $\theta : \text{Pic}(\mathbb{F}_q) \rightarrow \pi_1(X)^{\text{ab}}$ . It induces a map:

$$\theta^* : \{W_n^\times\text{-characters of } \pi_1(X)\} \rightarrow \{\text{morphisms } \text{Pic}(\mathbb{F}_q) \rightarrow W_n^\times\}.$$

Using  $\mathbb{L}_n$ , we will reconstruct the part of  $\theta^*$  on  $W_n^\times$ -characters which are trivial on  $\pi_1(x_0)$ .

2.5.2. Given a character  $\rho : \pi_1(X) \rightarrow W_n^\times$  trivial on  $\pi_1(x_0)$ , we will construct a character sheaf  $\text{Aut}_\rho \in W_n\text{-Shv}_c(\text{Jac})$  equipped with a canonical isomorphism (i.e., the Hecke eigen-property):

$$\text{add}^! \text{Aut}_\rho \xrightarrow{\sim} \rho \boxtimes \text{Aut}_\rho[1], \quad (2.13)$$

where  $\text{add}$  denotes the morphism:

$$\text{add} : X \times \text{Jac} \rightarrow \text{Jac}, \quad (x, \mathcal{L}) \rightsquigarrow \mathcal{L}(x - x_0).$$

Translation by  $\mathcal{O}(x_0) \in \text{Pic}^1(\mathbb{F}_q)$  then produces a character sheaf on  $\text{Pic}$  (still denoted by  $\text{Aut}_\rho$ ), and we will set  $\theta^* \rho$  to be the trace of Frobenius on  $\text{Aut}_\rho$  at each  $\mathbb{F}_q$ -point of  $\text{Pic}$ .

2.5.3. By Katz's equivalence (Lemma 1.4 for  $A = W_n$ ),  $\rho$  defines a line bundle  $\mathcal{L}_\rho$  over  $X_n$  with a connection, together with an isomorphism  $\text{Fr}_{X_n}^* \mathcal{L}_\rho \xrightarrow{\sim} \mathcal{L}_\rho$  and a rigidification of these data at  $x_0$ . Write  $\mathcal{L}_{\rho^{-1}}$  for its dual, so that  $\text{Sol}(\mathcal{L}_{\rho^{-1}}) \xrightarrow{\sim} \rho[1]$  under the solution functor of Emerton–Kisin (see §1.2.10). It gives rise to a  $W_n$ -point of  $\widetilde{\text{Jac}}_n^{\text{h}}$ :

$$i_{\rho^{-1}} : \text{Spec}(W_n) \rightarrow \widetilde{\text{Jac}}_n^{\text{h}}.$$

When equipped with the *identity* automorphism,  $(i_{\rho^{-1}})_* \mathcal{O}$  can be regarded as an object in  $\text{Coh}(\widetilde{\text{Jac}}_n^{\text{h}})^{\text{id}}$ . Define  $\text{Aut}_\rho$  as its image under  $\mathbb{L}_n$ . In other words, it is the solution complex of the object:

$$\mathcal{E}_{\rho^{-1}} \in \mathcal{D}^{(0)}\text{-Mod}(\text{Jac}_n)_{\text{Ifg-u,o}}^{\text{Fr}^*}$$

attached to  $(i_{\rho^{-1}})_* \mathcal{O}$ .

2.5.4. We now show that  $\text{Aut}_\rho$  is a character sheaf and construct the isomorphism (2.13). This latter will arise from an isomorphism:

$$\text{add}_n^*(\mathcal{E}_{\rho^{-1}}) \xrightarrow{\sim} \mathcal{L}_{\rho^{-1}} \boxtimes \mathcal{E}_{\rho^{-1}} \quad (2.14)$$

as  $\mathcal{D}^{(0)}$ -modules over  $X_n \times_{W_n} \text{Jac}_n$  together with a commutative diagram:

$$\begin{array}{ccc} \text{add}_n^*(\mathcal{E}_{\rho^{-1}}) & \longrightarrow & \mathcal{L}_{\rho^{-1}} \boxtimes \mathcal{E}_{\rho^{-1}} \\ \downarrow \text{add}_n^* \varphi_{\mathcal{E}_{\rho^{-1}}} & & \downarrow \varphi_{\mathcal{L}_{\rho^{-1}}} \boxtimes \varphi_{\mathcal{E}_{\rho^{-1}}} \\ \text{Fr}_{\text{Jac}_n}^* \text{add}_n^*(\mathcal{E}_{\rho^{-1}}) & \longrightarrow & \text{Fr}_{X_n}^* \mathcal{L}_{\rho^{-1}} \boxtimes \text{Fr}_{\text{Jac}_n}^* \mathcal{E}_{\rho^{-1}} \end{array} \quad (2.15)$$

Here, the maps  $\varphi_{\mathcal{E}_{\rho^{-1}}}$  and  $\varphi_{\mathcal{L}_{\rho^{-1}}}$  are the structural maps of the respective Frobenius  $\mathcal{D}^{(0)}$ -modules.

**Remark 2.12.** The operation that we call  $*$ -pullback on  $\mathcal{D}^{(0)}$ -modules passes to  $!$ -pullback of étale  $W_n$ -sheaves. This explains the notational difference between (2.13) and (2.14).

2.5.5. Indeed, the Fourier–Mukai–Laumon transform of  $(i_{\rho^{-1}})_* \mathcal{O}$  is a multiplicative object in  $\mathcal{D}^{(0)}\text{-Mod}(\text{Jac}_n)$  whose pullback along the Abel–Jacobi map:

$$X_n \rightarrow \text{Jac}_n, \quad x \rightsquigarrow \mathcal{O}(x - x_0)$$

identifies with  $\mathcal{L}_{\rho^{-1}}$ . One thus obtains (2.14) from the factorization of  $\text{add}_n$  as:

$$X_n \times_{W_n} \text{Jac}_n \xrightarrow{\text{AJ} \times \text{id}} \text{Jac}_n \times_{W_n} \text{Jac}_n \xrightarrow{m} \text{Jac}_n.$$

The commutative diagram (2.15) is a consequence of the functoriality of the construction.

### 3. CRITERION OF FULLY FAITHFULNESS

In this section, we prove our main result concerning the behavior of the  $p$ -torsion Artin reciprocity functor  $\mathbb{L}_n$ . Namely, it is fully faithful if and only if the Hasse–Witt matrix of  $X$  is nilpotent.

#### 3.1. Main result.

3.1.1. We remain in the setting  $k = \mathbb{F}_q$ . Let  $X$  be a smooth, proper, geometrically connected curve over  $\mathbb{F}_q$ , equipped with a smooth lift  $X_n$  to the ring  $W_n$ . In this setting, we have defined the reciprocity functor  $\mathbb{L}_n$  (see §2.4).

3.1.2. Our main result on the behavior of  $\mathbb{L}_n$  is as follows.

**Theorem 3.1.** *The following are equivalent:*

- (a) *The functor  $\mathbb{L}_n$  is fully faithful;*
- (b) *The endomorphism  $\text{Fr}_X^*$  on  $H^1(X, \mathcal{O}_X)$  is nilpotent.*

The Frobenius endomorphism on  $H^1(X, \mathcal{O}_X)$  is known as the Hasse–Witt matrix of the curve  $X$ . For example, when  $X$  is an elliptic curve, the endomorphism  $\text{Fr}_X^*$  vanishes if and only if  $X$  is supersingular.

#### 3.2. Reduction.

3.2.1. We first relate condition (b) to the geometry of the embedding of the Verschiebung-fixed locus  $\widetilde{\text{Jac}}_n^{\natural} \hookrightarrow \widetilde{\text{Jac}}_n$  (see §2.1.6). Let  $N_{\widetilde{\text{Jac}}_n^{\natural}/\widetilde{\text{Jac}}_n}$  denote the conormal sheaf of this closed immersion. Since the Verschiebung endomorphism of  $\widetilde{\text{Jac}}_n$  induces the identity map on  $\widetilde{\text{Jac}}_n^{\natural}$ , it defines an endomorphism of  $N_{\widetilde{\text{Jac}}_n^{\natural}/\widetilde{\text{Jac}}_n}$ .

**Lemma 3.2.** *Condition (b) is equivalent to:*

- (b') *The endomorphism on  $N_{\widetilde{\text{Jac}}_n^{\natural}/\widetilde{\text{Jac}}_n}$  induced by the Verschiebung is nilpotent.*

*Proof.* The proof consists of two parts. In the first part, we prove that (b) is equivalent to the assertion that the endomorphism on the tangent bundle  $T_{\widetilde{\text{Jac}}_n/W_n}|_{\widetilde{\text{Jac}}_n^{\natural}}$  induced from the Verschiebung is nilpotent. The latter endomorphism is nilpotent if and only if it is so on the special fiber  $T_{\widetilde{\text{Jac}}_1/\mathbb{F}_q}|_{\widetilde{\text{Jac}}_1^{\natural}}$ . Using the group structure of  $\widetilde{\text{Jac}}_1$  and the fact that  $\widetilde{\text{Jac}}_1^{\natural}$  is zero-dimensional (c.f. Lemma 2.2), this condition translates to nilpotence of the Verschiebung on  $T_{\widetilde{\text{Jac}}_1/\mathbb{F}_q}|_e$ , where  $e$  is the identity element of  $\widetilde{\text{Jac}}_1$ . There is an exact sequence of  $\mathbb{F}_q$ -vector spaces:

$$0 \rightarrow H^0(\Omega_{X/\mathbb{F}_q}) \rightarrow T_{\widetilde{\text{Jac}}_1/\mathbb{F}_q}|_e \xrightarrow{\pi} T_{\text{Jac}_1/\mathbb{F}_q}|_e \rightarrow 0.$$

Note that the map induced by the Verschiebung is zero on  $H^0(\Omega_{X/\mathbb{F}_q})$ ; on the other hand, we have a commutative diagram:

$$\begin{array}{ccc} T_{\text{Jac}_1/\mathbb{F}_q}|_e & \xrightarrow{\sim} & H^1(X, \mathcal{O}_X) \\ \downarrow \text{Ver} & & \downarrow \text{Fr}_X^* \\ T_{\widetilde{\text{Jac}}_1/\mathbb{F}_q}|_e & \xrightarrow{\sim} & H^1(X, \mathcal{O}_X) \end{array}$$

This shows that the condition (b) is equivalent to the nilpotence of the Verschiebung endomorphism on  $T_{\widetilde{\text{Jac}}_n/W_n}|_{\widetilde{\text{Jac}}_n^{\natural}}$ . Next, we will construct an isomorphism between  $N_{\widetilde{\text{Jac}}_n^{\natural}/\widetilde{\text{Jac}}_n}$  and the restriction of the cotangent bundle  $\Omega_{\widetilde{\text{Jac}}_n/W_n}|_{\widetilde{\text{Jac}}_n^{\natural}}$  which intertwines the Verschiebung endomorphism. This will complete the proof of the Lemma.

For this purpose, we consider a more abstract setting:  $S$  is a Noetherian base scheme,  $X \rightarrow S$  a smooth scheme,  $v : X \rightarrow X$  is an endomorphism,  $X^v$  the derived fixed point subscheme:

$$\begin{array}{ccc} X^v & \longrightarrow & X \\ \downarrow & & \downarrow \Delta \\ X & \xrightarrow{(v, \text{id})} & X \times_S X \end{array} \quad (3.1)$$

Suppose  $X^v \rightarrow X$  is a regular closed immersion of classical schemes. (In our context, this hypothesis is verified in Lemma 2.2.) Hence the cotangent complex  $L_{X^v/X}$  is quasi-isomorphic to  $N_{X^v/X}[1]$ . On the other hand, since (3.1) is a derived fiber product,  $L_{X^v/X}$  is identified with the restriction  $L_{X/(X \times_S X)}|_{X^v}$  where the immersion  $X \rightarrow X \times_S X$  is the diagonal map. The smoothness of  $X \rightarrow S$  yields a canonical identification of  $L_{X/(X \times_S X)}$  with  $\Omega_{X/S}[1]$ , so we find an isomorphism  $N_{X^v/X} \xrightarrow{\sim} \Omega_{X/S}|_{X^v}$ . It is straightforward to check that this isomorphism intertwines the endomorphisms induced from  $v$ .  $\square$

3.2.2. Let us now inspect condition (a). By construction of the functor  $\mathbb{L}_n$ , it is fully faithful if and only if the functor (2.11):

$$\text{Coh}(\widetilde{\text{Jac}}_n^{\natural})^{\text{id}} \rightarrow \text{Coh}(\widetilde{\text{Jac}}_n)^{\text{Ver}^*}$$

is fully faithful. Let us again consider a more abstract setting. Suppose  $S$  is a Noetherian base scheme and  $\iota : Y \hookrightarrow X$  is a closed immersion of  $S$ -schemes. Let  $v : X \rightarrow X$  be an endomorphism which induces the identity map on  $Y$  (i.e.,  $v\iota = \iota$ ). In this context, it is possible to consider the functor:

$$\text{Coh}(Y)^{\text{id}} \rightarrow \text{Coh}(X)^{v_*}, \quad (3.2)$$

sending  $(\mathcal{F}, \alpha)$  to the coherent sheaf  $\iota_*\mathcal{F}$  together with the isomorphism:

$$\iota_*\mathcal{F} \xrightarrow{\iota_*\alpha} \iota_*\mathcal{F} \xrightarrow{\sim} v_*\iota_*\mathcal{F}. \quad (3.3)$$

This construction includes (2.11) as a special case. Therefore, Theorem 3.1 is a consequence of the following general result.

**Proposition 3.3.** *Suppose  $\iota : Y \hookrightarrow X$  is a regular closed immersion of Noetherian schemes and  $Y$  is affine. Suppose  $v : X \rightarrow X$  is an endomorphism which induces the identity map on  $Y$ . Then the following are equivalent:*

- (a) *the functor (3.2) is fully faithful;*
- (b) *the endomorphism on  $N_{Y/X}$  induced by  $v$  is nilpotent.*

The rest of this section is devoted to the proof of Proposition 3.3. It will follow from an analysis of the Koszul resolution of  $\iota_*\mathcal{O}_Y$ .

### 3.3. Proof of Proposition 3.3.

3.3.1. Throughout this subsection, we put ourselves in the context of Proposition 3.3. Our first goal is to reformulate the fully faithfulness in more explicit terms. For the purpose of the proof, we need to consider lax-fixed points (see §1.3.2). Let  $(\mathcal{F}, \alpha) \in \text{Coh}(Y)^{\text{id} \rightarrow \text{id}}$  and  $(\mathcal{G}, \beta) \in \text{Coh}(Y)^{\text{id}}$ . Their Hom-space in  $\text{Coh}(Y)^{\text{id} \rightarrow \text{id}}$  is calculated as the fiber of the map:

$$\begin{array}{c} \text{Hom}_{\text{Coh}(Y)}(\mathcal{F}, \mathcal{G}) \\ \downarrow \text{id} - \beta^{-1} \cdot (-) \cdot \alpha \\ \text{Hom}_{\text{Coh}(Y)}(\mathcal{F}, \mathcal{G}) \end{array}$$

(see Lemma 1.9). On the other hand, the Hom-space of their images in  $\text{Coh}(X)^{\text{id} \rightarrow v_*}$  is calculated as the fiber of the map:

$$\begin{array}{c} \text{Hom}_{\text{Coh}(X)}(\iota_*\mathcal{F}, \iota_*\mathcal{G}) \\ \downarrow \text{id} - (\iota_*\beta)^{-1} \cdot v_*(-) \cdot \iota_*\alpha \\ \text{Hom}_{\text{Coh}(X)}(\iota_*\mathcal{F}, \iota_*\mathcal{G}) \end{array} \quad (3.4)$$

Here, the notation  $\iota_*\alpha$  (resp.  $\iota_*\beta$ ) is slightly abused to denote the composition (3.3).

3.3.2. It is possible to rewrite (3.4) as an endomorphism of  $\text{Hom}_{\text{Coh}(Y)}(\iota^*\iota_*\mathcal{F}, \mathcal{G})$  using the adjunction between  $\iota^*$  and  $\iota_*$ .<sup>7</sup> More precisely, for any  $\mathcal{F} \in \text{Coh}(Y)$ , we define a canonical endomorphism on  $\iota^*\iota_*\mathcal{F}$ :

$$v_{\mathcal{F}} : \iota^*\iota_*\mathcal{F} \rightarrow \iota^*\iota_*\mathcal{F}, \quad (3.5)$$

given by composing the canonical identification  $\iota^*\iota_*\mathcal{F} \xrightarrow{\sim} \iota^*v_*\iota_*\mathcal{F}$  with the map:

$$\iota^*v_*\mathcal{E} \rightarrow \iota^*\mathcal{E},$$

defined naturally for any quasi-coherent sheaf  $\mathcal{E}$  on  $X$  by adjunction and the equality  $v\iota = \iota$ , and which is then applied to  $\mathcal{E} = \iota_*\mathcal{F}$ . By a straightforward (but tedious) calculation, the morphism (3.4) is isomorphic to:

$$\begin{array}{c} \text{Hom}_{\text{Coh}(Y)}(\iota^*\iota_*\mathcal{F}, \mathcal{G}) \\ \downarrow \text{id} - \beta^{-1} \cdot (-) \cdot v_{\mathcal{F}} \cdot \iota^*\iota_*\alpha \\ \text{Hom}_{\text{Coh}(Y)}(\iota^*\iota_*\mathcal{F}, \mathcal{G}) \end{array}$$

3.3.3. In conclusion, the full faithfulness of (3.2) is equivalent to the Cartesian-ness of the following commutative square:

$$\begin{array}{ccc} \text{Hom}_{\text{Coh}(Y)}(\mathcal{F}, \mathcal{G}) & \longrightarrow & \text{Hom}_{\text{Coh}(Y)}(\iota^*\iota_*\mathcal{F}, \mathcal{G}) \\ \downarrow \text{id} - \beta^{-1} \cdot (-) \cdot \alpha & & \downarrow \text{id} - \beta^{-1} \cdot (-) \cdot v_{\mathcal{F}} \cdot \iota^*\iota_*\alpha \\ \text{Hom}_{\text{Coh}(Y)}(\mathcal{F}, \mathcal{G}) & \longrightarrow & \text{Hom}_{\text{Coh}(Y)}(\iota^*\iota_*\mathcal{F}, \mathcal{G}) \end{array} \quad (3.6)$$

when  $(\mathcal{F}, \alpha)$ ,  $(\mathcal{G}, \beta)$  both belong to  $\text{Coh}(Y)^{\text{id}}$ . We will use this formulation of condition (a) in our proof of Proposition 3.3. Here is a roadmap of the proof to follow.

- (a) We will first calculate  $v_{\mathcal{F}}$  for  $\mathcal{F} = \mathcal{O}_Y$ .
- (b) Next, we will prove “(a)  $\Rightarrow$  (b)” using the assumption on the Cartesian-ness of (3.6) only for  $(\mathcal{F}, \alpha) = (\mathcal{O}_Y, \text{id})$ ; this part of the proof will be completed in §3.3.5.
- (c) Finally, we will prove “(b)  $\Rightarrow$  (a)” by first establishing it for  $\mathcal{F}$  being finite locally free (essentially using the calculation of  $v_{\mathcal{F}}$  for  $\mathcal{F} = \mathcal{O}_Y$ ) and then reduces to this case by taking “free resolutions” of an arbitrary  $(\mathcal{F}, \alpha)$ ; the proof will be completed in §3.3.6.

**Lemma 3.4.** *There is a canonical isomorphism of  $\mathcal{O}_Y$ -modules for each  $i \geq 0$ :*

$$\bigwedge^i N_{Y/X} \xrightarrow{\sim} \text{H}^{-i}(\iota^*\iota_*\mathcal{O}_Y), \quad (3.7)$$

<sup>7</sup>Notation:  $\iota^*$  is the *derived* inverse image functor.

which renders the following diagram commutative

$$\begin{array}{ccc} \bigwedge^i N_{Y/X} & \xrightarrow{\cong} & H^{-i}(\iota^* \iota_* \mathcal{O}_Y) \\ \downarrow \bigwedge^i v & & \downarrow H^{-i} v_{\mathcal{O}_Y} \\ \bigwedge^i N_{Y/X} & \xrightarrow{\cong} & H^{-i}(\iota^* \iota_* \mathcal{O}_Y) \end{array} \quad (3.8)$$

*Proof.* Let  $\mathcal{J}$  denote the ideal sheaf of  $Y$ . The exact sequence:

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{O}_X \rightarrow \iota_* \mathcal{O}_Y \rightarrow 0 \quad (3.9)$$

induces an isomorphism between  $H^{-1} \iota^* \iota_* \mathcal{O}_Y$  and  $H^0 \iota^* \mathcal{J}$ , which is by definition  $N_{Y/X}$ . In other words, we have found the isomorphism  $N_{Y/X} \xrightarrow{\sim} H^{-1}(\iota^* \iota_* \mathcal{O}_Y)$ . This map renders (3.8) commutative for  $i = 1$ . The maps (3.7) for  $i \neq 1$  are constructed using the graded commutative algebra structure on  $H^\bullet(\iota^* \iota_* \mathcal{O}_Y)$ . The fact that they define isomorphisms making (3.8) commutative follows from the Koszul resolution of  $\iota_* \mathcal{O}_Y$  as an  $\mathcal{O}_X$ -algebra.  $\square$

3.3.4. We will now turn to a key step in the proof of Proposition 3.3. Namely, we will show that the fully faithfulness statement applied to  $(\mathcal{F}, \alpha) = (\mathcal{O}_Y, \text{id})$  implies the nilpotence of the  $v$ -action on  $N_{Y/X}$ . We begin with a commutative algebra fact.

**Lemma 3.5.** *Suppose  $R$  is a Noetherian ring and  $\mathcal{M}$  is a finite locally free  $R$ -module equipped with an endomorphism  $v$ . If for all finite  $R$ -module  $\mathcal{N}$  together with an automorphism  $\gamma$ , the endomorphism:*

$$\text{id} - \gamma \otimes v : \mathcal{N} \otimes \mathcal{M} \rightarrow \mathcal{N} \otimes \mathcal{M}$$

*is invertible, then  $v$  is nilpotent.*

*Proof.* Without loss of generality, we may assume that  $\text{Spec}(R)$  is connected and  $\mathcal{M}$  has constant rank  $r$ . We first show that  $v$  is *not* invertible. Suppose the contrary. Let  $\mathcal{N} := \mathcal{M}^\vee$  together with the automorphism  $\gamma := (v^{-1})^\vee$ . We have a commutative diagram:

$$\begin{array}{ccc} \mathcal{M}^\vee \otimes \mathcal{M} & \longrightarrow & \text{End}(\mathcal{M}) \\ \downarrow \text{id} - (v^{-1})^\vee \otimes v & & \downarrow \text{id} - v(-)v^{-1} \\ \mathcal{M}^\vee \otimes \mathcal{M} & \longrightarrow & \text{End}(\mathcal{M}) \end{array}$$

where the right vertical arrow annihilates  $\text{id}_{\mathcal{M}}$ . Contradiction. We will now prove that  $v$  is nilpotent for increasingly general rings  $R$ .

First, we suppose that  $R$  is a field. In this case,  $\mathcal{M}$  is a finite-dimensional  $R$ -vector space. Hence  $v$  is not injective. Let  $\mathcal{M}_1 \neq 0$  be its kernel and  $\mathcal{Q}_1 := \mathcal{M}/\mathcal{M}_1$  be the quotient. The endomorphism  $v$  induces an endomorphism  $v_1$  on  $\mathcal{Q}_1$ . For any  $(\mathcal{N}, \gamma)$  as in the statement of the Lemma, we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc} \mathcal{N} \otimes \mathcal{M}_1 & \longrightarrow & \mathcal{N} \otimes \mathcal{M} & \longrightarrow & \mathcal{N} \otimes \mathcal{Q}_1 & \longrightarrow & 0 \\ \downarrow \text{id} & & \downarrow \text{id} - \gamma \otimes v & & \downarrow \text{id} - \gamma \otimes v_1 & & \\ \mathcal{N} \otimes \mathcal{M}_1 & \longrightarrow & \mathcal{N} \otimes \mathcal{M} & \longrightarrow & \mathcal{N} \otimes \mathcal{Q}_1 & \longrightarrow & 0 \end{array}$$

Since the middle arrow is invertible, so is the right arrow. Hence  $(\mathcal{Q}_1, v_1)$  again satisfies the hypothesis of the Lemma. This shows that we may inductively build a filtration:

$$0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots \subset \mathcal{M},$$

where  $v(\mathcal{M}_i) \subset \mathcal{M}_{i-1}$ . Since  $\mathcal{M}$  has finite dimension, this filtration terminates at finite step and we conclude that  $v$  is nilpotent.

Now, we suppose that  $R$  is reduced and Noetherian. For any  $\mathfrak{p} \in \text{Spec}(R)$ , the base change of  $(\mathcal{M}, v)$  to  $\kappa(\mathfrak{p})$  satisfies the hypothesis of the Lemma (with  $R$  replaced by  $\kappa(\mathfrak{p})$ ). Hence the previous step shows that  $v^r$  reduces to zero at each fiber  $R \rightarrow \kappa(\mathfrak{p})$  (where  $r$  is the rank of  $\mathcal{M}$ ). Let  $\mathcal{M}'$  denote the cokernel of the map

$$\mathcal{M} \xrightarrow{v^r} \mathcal{M} \rightarrow \mathcal{M}' \rightarrow 0.$$

Then  $\mathcal{M}'$  has constant fibers of dimension  $r$ . Since  $R$  is reduced, it follows that  $\mathcal{M}'$  is locally free of rank  $r$ . The surjection  $\mathcal{M} \rightarrow \mathcal{M}'$  must therefore be an isomorphism. Hence  $v^r = 0$ .

Finally, we suppose that  $R$  is any Noetherian ring. The previous step shows that the endomorphism  $v^r : \mathcal{M} \rightarrow \mathcal{M}$  factors through  $\mathfrak{n}\mathcal{M}$ , where  $\mathfrak{n}$  is the nilradical of  $R$ . Since  $R$  is Noetherian,  $\mathfrak{n}$  is nilpotent. Hence we must have  $v^{r'}(\mathcal{M}) = 0$  for some integer  $r' \geq r$ .  $\square$

3.3.5. The implication “(a)  $\Rightarrow$  (b)” in Proposition 3.3 is a consequence of the following.

**Lemma 3.6.** *If for every object  $(\mathcal{G}, \beta) \in \text{Coh}(Y)^{\text{id}}$ , the commutative square (3.6) for  $(\mathcal{F}, \alpha) = (\mathcal{O}_Y, \text{id})$  is Cartesian, then the  $v$ -action on  $N_{Y/X}$  is nilpotent.*

*Proof.* Since  $\mathcal{O}_Y$  is identified with  $H^0 \iota^* \iota_* \mathcal{O}_Y$  via the co-unit map, the assumption implies (is in fact equivalent to) that the endomorphism  $\text{id} - \beta^{-1} \cdot (-) \cdot \tau^{\leq -1} v_{\mathcal{O}_Y}$  acting on the space  $\text{Hom}_{\text{Coh}(Y)}(\tau^{\leq -1} \iota^* \iota_* \mathcal{O}_Y, \mathcal{G})$  is an equivalence. Taking  $\mathcal{G}$  to be any coherent sheaf  $\mathcal{N}$  concentrated in cohomological degree  $-1$ , this assertion implies that:

$$\text{id} - \beta^{-1} \cdot (-) \cdot v : \text{Hom}_{\text{Coh}(Y)^\heartsuit}(N_{Y/X}, \mathcal{N}) \rightarrow \text{Hom}_{\text{Coh}(Y)^\heartsuit}(N_{Y/X}, \mathcal{N})$$

is bijective; here we have used Lemma 3.4 to identify  $H^{-1} \iota^* \iota_* \mathcal{O}_Y$  with  $N_{Y/X}$  in a way compatible with the  $v$ -action. Since:

$$\text{Hom}_{\text{Coh}(Y)^\heartsuit}(N_{Y/X}, \mathcal{N}) \xrightarrow{\sim} \mathcal{N} \otimes N_{Y/X}^\vee,$$

we may apply Lemma 3.5 to  $(\mathcal{M}, v) = (N_{Y/X}^\vee, v^\vee)$  to conclude.  $\square$

3.3.6. We now turn to the “(b)  $\Rightarrow$  (a)” direction in Proposition 3.3. Namely, we will assume that the endomorphism on  $N_{Y/X}$  induced from  $v$  is nilpotent. We will prove a stronger statement: the commutative square (3.6) is Cartesian for all  $(\mathcal{F}, \alpha) \in \text{Coh}(Y)^{\text{id} \rightarrow \text{id}}$  and  $(\mathcal{G}, \beta) \in \text{Coh}(Y)^{\text{id}}$ .

**Lemma 3.7.** *Under the assumptions of §3.3.6, the commutative square (3.6) is Cartesian if  $\mathcal{F}$  is a finite locally free  $\mathcal{O}_Y$ -module.*

*Proof.* Since the problem is local on  $Y$ , we may assume that  $\mathcal{F}$  is a free  $\mathcal{O}_Y$ -module of rank  $k$ . Because  $\mathcal{O}_Y$  is identified with  $H^0 \iota^* \iota_* \mathcal{O}_Y$  via the co-unit map, it suffices to show that the endomorphism:

$$\begin{array}{c} \text{Hom}_{\text{Coh}(Y)}(\tau^{\leq -1} \iota^* \iota_* \mathcal{F}, \mathcal{G}) \\ \downarrow \text{id} - \beta^{-1} \cdot (-) \cdot \tau^{\leq -1} v_{\mathcal{F}} \cdot \tau^{\leq -1} \iota^* \iota_* \alpha \\ \text{Hom}_{\text{Coh}(Y)}(\tau^{\leq -1} \iota^* \iota_* \mathcal{F}, \mathcal{G}) \end{array} \quad (3.10)$$

is an isomorphism. Since  $\iota^* \iota_* \mathcal{F}$  is cohomological bounded, via cohomological truncations, it suffices to prove that for each  $i \geq 1$ , the following map:

$$\begin{array}{c} \text{Hom}_{\text{Coh}(Y)}(H^{-i}(\iota^* \iota_* \mathcal{F}), \mathcal{G}) \\ \downarrow \text{id} - \beta^{-1} \cdot (-) \cdot H^{-i} v_{\mathcal{F}} \cdot H^{-i} \iota^* \iota_* \alpha \\ \text{Hom}_{\text{Coh}(Y)}(H^{-i}(\iota^* \iota_* \mathcal{F}), \mathcal{G}) \end{array} \quad (3.11)$$

is invertible. Since  $v_{\mathcal{F}}$  is functorially assigned to  $\mathcal{F}$ , the endomorphisms  $v_{\mathcal{F}}$  and  $\iota^* \iota_* \alpha$  commute. Thus, so do their truncated versions. It follows that the  $n$ -fold composition of  $\beta^{-1} \cdot (-) \cdot H^{-i} v_{\mathcal{F}} \cdot H^{-i} \iota^* \iota_* \alpha$  is given by:

$$\beta^{-n} \cdot (-) \cdot (H^{-i} v_{\mathcal{F}})^n \cdot (H^{-i} \iota^* \iota_* \alpha)^n.$$

Since  $H^{-i} v_{\mathcal{F}}$  is identified with  $H^{-i} v_{\mathcal{O}_Y}^{\oplus k}$  acting on  $H^{-i} \iota^* \iota_* \mathcal{O}_Y^{\oplus k}$ , the hypothesis on nilpotence and Lemma (3.4) shows that  $(H^{-i} v_{\mathcal{F}})^n$  (for  $i \geq 1$ ) vanishes for  $n$  sufficiently large. This implies that (3.11) is invertible.  $\square$

3.3.7. Finally, we will reduce the general statement to the case where  $\mathcal{F}$  is a finite locally free  $\mathcal{O}_Y$ -module. For this reduction, it is convenient to introduce the cocomplete  $\infty$ -category:

$$\mathrm{QCoh}(Y)_{\mathrm{loc. fin}}^{\mathrm{id} \rightarrow \mathrm{id}} \subset \mathrm{QCoh}(Y)^{\mathrm{id} \rightarrow \mathrm{id}},$$

consisting of objects  $(\mathcal{M}, \beta)$  such that for each  $i$ , the action of  $H^i(\beta)$  on  $H^i(\mathcal{M})$  is *locally finite*, i.e.,  $H^i(\mathcal{M})$  is a union of finite submodules stable under  $H^i(\beta)$ . It is clear that the full subcategory  $\mathrm{Coh}(Y)^{\mathrm{id} \rightarrow \mathrm{id}} \subset \mathrm{QCoh}(Y)^{\mathrm{id} \rightarrow \mathrm{id}}$  is contained in the ‘‘locally finite’’ subcategory.

**Lemma 3.8.** *The stable  $\infty$ -category  $\mathrm{QCoh}(Y)_{\mathrm{loc. fin}}^{\mathrm{id} \rightarrow \mathrm{id}}$  is generated under colimits by objects of the form  $(\mathcal{F}, \alpha)$  where  $\mathcal{F}$  is a finite locally free  $\mathcal{O}_Y$ -module up to cohomological shift.*

*Proof.* It suffices to show that every nonzero object  $(\mathcal{M}, \beta)$  of  $\mathrm{QCoh}(Y)_{\mathrm{loc. fin}}^{\mathrm{id} \rightarrow \mathrm{id}}$  receives a nonzero map from some  $(\mathcal{F}, \alpha)$  as in the statement of the Lemma. Suppose  $H^i(\mathcal{M}) \neq 0$  for some  $i$ . We will construct such a pair  $(\mathcal{F}, \alpha)$  equipped with a map to  $(\tau^{\leq i} \mathcal{M}, \tau^{\leq i} \beta)$  such that the induced map on  $H^i$  is nonzero. The  $\infty$ -category  $\mathrm{QCoh}(Y)^{< \infty}$  is the differential graded nerve of bounded above complexes of projective modules (c.f. [Lu17, 1.3.2.7]). Hence we may assume that  $\tau^{\leq i} \mathcal{M}$  is such a complex and  $\beta$  is represented by a chain map. Hence  $(\tau^{\leq i} \mathcal{M}, \tau^{\leq i} \beta)$  is of the form:

$$\begin{array}{ccccccc} \dots & \longrightarrow & M^{i-2} & \xrightarrow{\delta^{i-2}} & \mathcal{M}^{i-1} & \xrightarrow{\delta^{i-1}} & \mathrm{Ker}(\delta^i) \longrightarrow 0 \longrightarrow \dots \\ & & \downarrow \beta^{i-2} & & \downarrow \beta^{i-1} & & \downarrow \beta^i \\ \dots & \longrightarrow & M^{i-2} & \xrightarrow{\delta^{i-2}} & \mathcal{M}^{i-1} & \xrightarrow{\delta^{i-1}} & \mathrm{Ker}(\delta^i) \longrightarrow 0 \longrightarrow \dots \end{array}$$

where each  $\mathcal{M}^j$  is projective. Since the action of  $H^i(\beta)$  on  $H^i(\mathcal{M})$  is locally finite, we can find a finite submodule  $\mathcal{N} \subset H^i(\mathcal{M})$  stable under  $H^i(\beta)$ . Then there exists a surjection from some finite locally free  $\mathcal{O}_Y$ -module  $f : \mathcal{F} \rightarrow \mathcal{N}$ , and an endomorphism  $\alpha$  of  $\mathcal{F}$  making the following diagram commute:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{f} & \mathcal{N} \\ \downarrow \alpha & & \downarrow H^i(\beta) \\ \mathcal{F} & \xrightarrow{f} & \mathcal{N} \end{array}$$

Choose any lift of  $f$  to a map  $\tilde{f} : \mathcal{F} \rightarrow \mathrm{Ker}(\delta^i)$ . Then  $\tilde{f}$  defines a chain map  $\mathcal{F}[-i] \rightarrow \tau^{\leq i} \mathcal{M}$ . Furthermore, the two maps  $\beta^i \tilde{f}$  and  $\tilde{f} \alpha$  from  $\mathcal{F}$  to  $\mathrm{Ker}(\delta^i)$  differ by a morphism  $\mathcal{F} \rightarrow \mathrm{Im}(\delta^{i-1})$ , which can be lifted to a map  $h : \mathcal{F} \rightarrow \mathcal{M}^{i-1}$ . The map  $h$  witnesses the commutativity of the following diagram in  $\mathrm{QCoh}(Y)^{< \infty}$ :

$$\begin{array}{ccc} \mathcal{F}[-i] & \longrightarrow & \tau^{\leq i} \mathcal{M} \\ \downarrow \alpha[-i] & & \downarrow \tau^{\leq i} \beta \\ \mathcal{F}[-i] & \longrightarrow & \tau^{\leq i} \mathcal{M} \end{array}$$

Hence we find a morphism from  $(\mathcal{F}, \alpha)[-i]$  to  $(\tau^{\leq i} \mathcal{M}, \tau^{\leq i} \beta)$  in  $\mathrm{QCoh}(Y)_{\mathrm{loc. fin}}^{\mathrm{id} \rightarrow \mathrm{id}}$  which induces a nonzero map on  $H^i$ .  $\square$

3.3.8. To prove “(b)  $\Rightarrow$  (a)” in Proposition 3.3, one observes that the commutative square associated to  $(\mathcal{F}, \alpha) \in \mathrm{QCoh}(Y)_{\mathrm{loc. fin}}^{\mathrm{id} \rightarrow \mathrm{id}}$  and  $(\mathcal{G}, \beta) \in \mathrm{Coh}(Y)^{\mathrm{id}}$ :

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{QCoh}(Y)}(\mathcal{F}, \mathcal{G}) & \longrightarrow & \mathrm{Hom}_{\mathrm{QCoh}(Y)}(\iota^* \iota_* \mathcal{F}, \mathcal{G}) \\ \downarrow \mathrm{id} - \beta^{-1} \cdot (-) \cdot \alpha & & \downarrow \mathrm{id} - \beta^{-1} \cdot (-) \cdot v_{\mathcal{F}} \cdot \iota^* \iota_* \alpha \\ \mathrm{Hom}_{\mathrm{QCoh}(Y)}(\mathcal{F}, \mathcal{G}) & \longrightarrow & \mathrm{Hom}_{\mathrm{QCoh}(Y)}(\iota^* \iota_* \mathcal{F}, \mathcal{G}) \end{array}$$

defined analogously to (3.6), takes colimits in the object  $(\mathcal{F}, \alpha) \in \mathrm{QCoh}(Y)_{\mathrm{loc. fin}}^{\mathrm{id} \rightarrow \mathrm{id}}$  to limits of commutative squares. Hence the problem reduces to the case where  $\mathcal{F}$  is a finite locally free  $\mathcal{O}_Y$ -module up to cohomological shift, where it follows from Lemma 3.7.

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