SPECTRAL DECOMPOSITION OF GENUINE CUSP FORMS OVER GLOBAL FUNCTION FIELDS

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Abstract. We prove the geometric Satake equivalence for étale metaplectic covers of reductive group schemes and extend the Langlands parametrization of V. Lafforgue to genuine cusp forms defined on their associated covering groups.

Contents

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Introduction

This article is a contibution to the Langlands program for covering groups, as proposed by Weissman, Gan, and Gao [Wei18] [GG18]. Its goal is to parametrize genuine cusp forms over a global function field by spectral data, defined in terms of an L-group.

Such a parametrization has been anticipated by V. Lafforgue [Laf18, §14] and Gan–Gao [GG18, §14, Question (L)]. V. Lafforgue has moreover indicated a path towards it via the arguments of *op.cit.*, combined with a strong version of the geometric Satake equivalence for covering groups, which in principle should follow from Finkelberg–Lysenko [FL10], Reich [Rei12], and Gaitsgory–Lysenko [GL18].

The present article is intended to realize this vision in what we believe is its appropriate generality. More concretely, the class of covering groups treated in this article includes the ones defined by Brylinski–Deligne [BD01] using algebraic K-theory, but generally contains more objects when the reductive group is not simply connected. In this sense, our scope is larger than the one envisioned by Weissman [Wei18] and Gan–Gao [GG18].

By making this generalization, we also make the problem *simpler*, essentially because étale cohomology is better understood than algebraic K-theory. This alternative perspective goes back to Deligne [Del96] and is rediscovered by Gaitsgory–Lysenko [GL18] under a different guise. To keep our narrative coherent, we use [Zha22] as our only input concerning covering groups, although many results proved there have analogues in [GL18].

0.1. Main result.

0.1.1. Let F be a global field of characteristic $p \neq 0$. Denote by $\mathbb{A}_{\mathrm{F}}$ the topological ring of its adèles.

Let $\ell \neq p$ be a prime and choose an algebraic closure $\overline{\mathbb{Q}}_\ell \supset \mathbb{Q}_\ell$. Let $\mathrm{A} \subset \overline{\mathbb{Q}}_\ell^\times$ be a finite subgroup whose order is indivisible by p.

Our group-theoretic input is a pair (G, μ), where G is a reductive group over F and μ is a rigidified section of the (higher) étale stack $\mathrm{B}^4\mathrm{A}(1)$ over the classifying stack BG, *i.e.* a section equipped with a trivialization along the unit $e : \mathrm{Spec}(\mathrm{F}) \to \mathrm{BG}$.

The datum μ is called an *étale metaplectic cover* of G in [Zha22]. It categorifies a class in the reduced étale cohomology group $\mathrm{H}^4_e(\mathrm{BG}, \mathrm{A}(1))$. In the special case where G is simply connected, the space of étale metaplectic covers is discrete, and our formalism coincides with the one in [Del96].

0.1.2. From the pair (G, μ), we extract two pieces of "classical" structures. (To construct them, it is essential to start with μ rather than the cohomology class it represents.)

The first one is a central extension of topological groups:

$$1 \to \mathrm{A} \to \widetilde{\mathrm{G}}_{\mathrm{F}} \to \mathrm{G}(\mathbb{A}_{\mathrm{F}}) \to 1, \tag{0.1}$$

equipped with a canonical splitting over $\mathrm{G}(\mathrm{F}) \subset \mathrm{G}(\mathbb{A}_{\mathrm{F}})$. The central extension (0.1) gives us the notion of a *genuine automorphic form*: a $\mathrm{G}(\mathrm{F})$-invariant locally constant function $f : \widetilde{\mathrm{G}}_{\mathrm{F}} \to \overline{\mathbb{Q}}_\ell$ satisfying the equality $f(\tilde{x} \cdot a) = f(\tilde{x}) \cdot a$ for each $\tilde{x} \in \widetilde{\mathrm{G}}_{\mathrm{F}}$ and $a \in \mathrm{A}$.

The second piece of structure is a short exact sequence of topological groups:

$$1 \to \mathrm{H}_{\bar{\mathrm{F}}}(\overline{\mathbb{Q}}_\ell) \to {}^{\mathrm{L}}\mathrm{H}_{\mathrm{F},\vartheta} \to \mathrm{Gal}(\bar{\mathrm{F}}/\mathrm{F}) \to 1, \tag{0.2}$$

where $H_{\bar{F}}$ is a pinned split reductive group over $\overline{\mathbb{Q}}_\ell$. It is determined by (G, μ) following a combinatorial recipe, modulo some immaterial choices such as the algebraic closure $\bar{F}$. With (0.2), we arrive at the notion of an *L-parameter*: an $H_{\bar{F}}(\overline{\mathbb{Q}}_\ell)$-conjugacy class of sections $\sigma : \mathrm{Gal}(\bar{F}/F) \to {}^L H_{F,\vartheta}$ of (0.2).

Remark 0.1.3. For classical applications, one often takes A to be the group $\mu(F)$ of roots of unity in F, whose inclusion in $\overline{\mathbb{Q}}_\ell^\times$ is written as an injective character $\zeta : \mu(F) \to \overline{\mathbb{Q}}_\ell^\times$.

Given an integer $n \geq 1$ invertible in F, any Brylinski–Deligne extension of G induces an étale metaplectic cover with $A = \mu_n$ (see [Zha22, §2.3]). When $\mu_n(F)$ has cardinality n, the covering group (0.1) agrees with the one constructed in [BD01, §10], whereas the L-group (0.2) is identified with the one constructed by Weissman [Wei18].

Another source of étale metaplectic covers arises from morphisms of complexes $\pi_1(G) \to A[2]$ of $\mathrm{Gal}(\bar{F}/F)$-modules, where $\pi_1(G)$ denotes the algebraic fundamental group of G (see [Zha22, §5.3]). Over a p-adic local field, they induce covering groups of Kaletha [Kal22].

0.1.4. The main result of this article is a parametrization of the cuspidal part of genuine automorphic forms by L-parameters.

In order to formulate this parametrization, we need an additional piece of information having to do with the maximal torus Z of the center $Z_G \subset G$.

To wit, the covering group of $Z(\mathbb{A}_F)$ induced from (0.1) is generally not commutative. However, there is a canonically defined isogeny of tori $Z^\sharp \to Z$ such that the induced covering group $\widetilde{Z}^\sharp_F \to Z^\sharp(\mathbb{A}_F)$ is commutative and its image in $\widetilde{G}_F$ is central.

We shall fix a lattice (*i.e.* discrete and cocompact subgroup) $\Xi \subset \widetilde{Z}^\sharp_F / Z^\sharp(F)$ which projects isomorphically onto its image in $Z^\sharp(\mathbb{A}_F)/Z^\sharp(F)$.

Furthermore, the restriction of (0.1) to $P(\mathbb{A}_F)$, for each parabolic subgroup $P \subset G$, canonically descends to the Levi quotient $M(\mathbb{A}_F)$. We can thus define the *cuspidal part* of compactly supported genuine automorphic forms on $\widetilde{G}_F/G(F)\Xi$, by imposing the vanishing of constant terms for all proper parabolic subgroups $P \subset G$.

Our main theorem, in its most classical form, is a decomposition of this $\overline{\mathbb{Q}}_\ell$-vector space according to L-parameters.

Theorem A. *There is a canonical decomposition:*

$$\mathrm{Fun}_{\mathrm{cusp}}(\widetilde{G}_F/G(F)\Xi, A \subset \overline{\mathbb{Q}}_\ell^\times) \cong \bigoplus_{[\sigma]} \mathbf{H}_{[\sigma]}, \tag{0.3}$$

where $[\sigma]$ *ranges over* $H_{\bar{F}}(\overline{\mathbb{Q}}_\ell)$*-conjugacy classes of sections of* (0.2).

0.1.5. The spectral decomposition (0.3) arises as the limiting case of its integral variant, Theorem 4.3.11, which contains additional information such as the compatibility with the Satake isomorphism for covering groups, *c.f.* [McN12].

Taking this compatibility for granted, Theorem A fulfills the "automorphic-to-Galois" direction of Langlands reciprocity for genuine cusp forms. In the absence of covering groups, this result is established by Drinfeld for GL_2 [Dri87b] [Dri88] [Dri87a], L. Lafforgue for GL_n [Laf02], and V. Lafforgue for all reductive groups [Laf18].

0.2. Outline of the proof.

0.2.1. The proof of Theorem A is an adaptation of [Laf18]. First, we must formulate an integral version of the problem in order to use the tools of algebraic geometry.

Let X be smooth, proper, geometrically connected curve over a finite field k with generic point $\eta = \mathrm{Spec}(F)$. Let $D \subset X$ be a k-finite closed subscheme and $\mathring{X} := X \backslash D$ be its complement.

The notations $\overline{\mathbb{Q}}_\ell$ and A are as in §0.1.1.

We replace the group-theoretic input by a pair (G, μ), where $\mathrm{G} \to \mathrm{X}$ is a smooth affine group scheme whose base change to $\mathring{\mathrm{X}}$ is reductive and μ is a rigidified section of $\mathrm{B}^4\mathrm{A}(1)$ over $\mathrm{B}_{\mathring{\mathrm{X}}}(\mathrm{G})$, the classifying stack of the base change of G to $\mathring{\mathrm{X}}$.

0.2.2. Denote by $\mathrm{Bun}_{\mathrm{G},\mathrm{D}}$ the moduli stack of G-torsors over X rigidified along D.

The étale metaplectic cover μ defines an A-gerbe $\mathscr{G}_{\mathrm{D}}$ over $\mathrm{Bun}_{\mathrm{G},\mathrm{D}}$, via the categorification of a transgression map on étale cohomology:

$$[\mathrm{X}] : \mathrm{H}^4_e(\mathrm{BG}, \mathrm{A}(1)) \to \mathrm{H}^2(\mathrm{Bun}_{\mathrm{G},\mathrm{D}}, \mathrm{A}).$$

The A-gerbe $\mathscr{G}_{\mathrm{D}}$ geometrizes the covering group $\widetilde{\mathrm{G}}_{\mathrm{F}}$ in the sense that a process akin to taking the trace of Frobenius yields a set-theoretic A-torsor:

$$\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}} \to \mathrm{Bun}_{\mathrm{G},\mathrm{D}}(k), \tag{0.4}$$

whose pullback along the adèlic uniformation map $\mathrm{G}(\mathbb{A}_{\mathrm{F}}) \to \mathrm{Bun}_{\mathrm{G},\mathrm{D}}(k)$ recovers $\widetilde{\mathrm{G}}_{\mathrm{F}}$.

It is more natural, especially for nonsplit reductive groups, to replace genuine automorphic forms by A-equivariant functions on $\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}}$, and the spectral decomposition theorem will hold for this larger space of functions.

0.2.3. There are two other important geometric objects associated to G: the local Hecke stack and the moduli stack of Shtukas defined by Drinfeld [Dri87b] and Varshavsky [Var04].

For a nonempty finite set I, these objects are ind-algebraic stacks over $\mathring{\mathrm{X}}^{\mathrm{I}}$, related by a morphism defined as restriction to the parametrized formal disks:

$$\mathrm{res} : \mathrm{Sht}^{\mathrm{I}}_{\mathrm{G},\mathrm{D}} \to \mathrm{Hec}^{\mathrm{I}}_{\mathrm{G}}.$$

The rigidified section μ also defines an étale A-gerbe $\mathscr{G}^{\mathrm{I}}$ over $\mathrm{Hec}^{\mathrm{I}}_{\mathrm{G}}$, which geometrizes the local covering groups $\widetilde{\mathrm{G}}_x \to \mathrm{G}(\mathrm{F}_x)$ together with their canonical splittings over the maximal compact subgroups $\mathrm{G}(\mathscr{O}_x)$, for each $x \in \mathring{\mathrm{X}}$.

The key observation is that $\mathscr{G}^{\mathrm{I}}$ is canonically trivialized over $\mathrm{Sht}^{\mathrm{I}}_{\mathrm{G},\mathrm{D}}$. In particular, $\mathscr{G}^{\mathrm{I}}$-twisted ℓ-adic sheaves on $\mathrm{Hec}^{\mathrm{I}}_{\mathrm{G}}$ pull back to *untwisted* sheaves over $\mathrm{Sht}^{\mathrm{I}}_{\mathrm{G},\mathrm{D}}$, so their compactly supported cohomology are usual ℓ-adic sheaves over $\mathring{\mathrm{X}}^{\mathrm{I}}$.

Moreoever, $\mathscr{G}^{\mathrm{I}}$ is canonically trivialized over the unit section e of $\mathrm{Hec}^{\mathrm{I}}_{\mathrm{G}}$, so $e_!(\overline{\mathbb{Q}}_\ell)$ may be viewed as a $\mathscr{G}^{\mathrm{I}}$-twisted sheaf over $\mathrm{Hec}^{\mathrm{I}}_{\mathrm{G}}$. Applying the above process to $e_!(\overline{\mathbb{Q}}_\ell)$, we find the constant sheaf over $\mathring{\mathrm{X}}^{\mathrm{I}}$ with coefficients in compactly supported A-equivariant functions on $\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}}$. This is how cohomology of Shtukas encodes genuine automorphic forms.

Remark 0.2.4. In the main body of the text, these constructions will be applied to the quotient stack $\mathrm{Sht}^{\mathrm{I}}_{\mathrm{G},\mathrm{D}}/\Xi$, where $\Xi \subset \widetilde{\mathrm{Bun}}_{\mathrm{Z}^\sharp,\infty\mathrm{D}}$ is a lattice analogous to the one in §0.1.4, but let us ignore this difference for now.

0.2.5. Turning to the spectral side, we extract from the pair (G, μ) a locally constant étale sheaf over $\mathring{\mathrm{X}}$ of pinned split reductive groups H over $\overline{\mathbb{Q}}_\ell$, together with an $\mathbb{E}_\infty$-monoidal morphism:

$$\mathrm{F}_\vartheta : \hat{\mathrm{Z}}_{\mathrm{H}} \to \mathrm{B}^2_{\mathring{\mathrm{X}}}(\mathrm{A}), \tag{0.5}$$

where $\hat{\mathrm{Z}}_{\mathrm{H}}$ denotes the sheaf of characters of the center $\mathrm{Z}_{\mathrm{H}} \subset \mathrm{H}$.

The subscript ϑ in (0.5) refers to a twist by the $\{\pm 1\}$-gerbe of theta characteristics over $\mathring{\mathrm{X}}$ (relevant only when the characteristic $p \neq 2$). The somewhat curious Corollary 4.2.7 shows that it is essentially equivalent to Weissman's meta-Galois group.

The pair $(\mathrm{H}, \mathrm{F}_\vartheta)$ is our version of the *metaplectic dual data*. It is closely related to the same-named notion in [GL18], although we use the construction in [Zha22, §6] which is valid in the number field situation as well.

In the absence of an étale metaplectic cover, H would be the sheaf-theoretic version of Langlands' L-group associated to G. The object F_ϑ is particular to the metaplectic context, and can be concretely described as an extension of stacks of Picard groupoids over $\mathring{X}$:

$$B_{\mathring{X}}(A) \to F_\vartheta^\dagger \to \hat{Z}_H. \tag{0.6}$$

0.2.6. Here, we encounter an interesting phenomenon which is only visible on the geometric level: $F_\vartheta^\dagger$ is generally *not* strictly commutative. Equivalently put, (0.5) generally does not come from a morphism of complexes $\hat{Z}_H \to A[2]$ of sheaves of abelian groups.

However, we may modify the commutativity constraint on $F_\vartheta^\dagger$ in a canonical way to make it strictly commutative. This gives us a morphism of complexes:

$$^0F_\vartheta : \hat{Z}_H \to A[2]. \tag{0.7}$$

Inducing along $A \subset \overline{\mathbb{Q}}_\ell^\times$, (0.7) defines an étale $Z_H(\overline{\mathbb{Q}}_\ell)$-gerbe over $\mathring{X}$.

The L-group is a way to repackage the data $(H, {}^0F_\vartheta)$. Namely, if $D \neq \varnothing$, after choosing a geometric point $\bar{\eta} = \mathrm{Spec}(\bar{F}) \to \eta$ and a rigidification of ${}^0F_\vartheta$ along $\bar{\eta}$, we obtain a short exact sequence of topological groups:

$$1 \to H_{\bar{\eta}}(\overline{\mathbb{Q}}_\ell) \to {}^LH_{\mathring{X},\vartheta} \to \pi_1(\mathring{X}, \bar{\eta}) \to 1. \tag{0.8}$$

The generic form of the L-group (0.2) occurs as its pullback to $\pi_1(\eta, \bar{\eta})$, with notational change $H_{\bar{\eta}} = H_{\bar{F}}$.

0.2.7. As in [Laf18], the L-group enters through the geometric Satake equivalence.

In our context, this equivalence will make (H, F_ϑ) appear naturally. Indeed, we consider the stack of (finite-rank) H-representations on lisse $\overline{\mathbb{Q}}_\ell$-sheaves over $\mathring{X}$. This is an étale stack of tensor categories equipped with a grading by $\hat{Z}_H$.

The $\mathbb{E}_\infty$-monoidal morphism F_ϑ allows us to twist this étale stack, whose global section over $\mathring{X}$ is a new tensor category $\mathrm{Rep}_{H,F_\vartheta}$.

On the other hand, we consider the category $\mathrm{Sat}_{G,\mathscr{G}}$ of $\mathscr{G}$-twisted constructible complexes of $\overline{\mathbb{Q}}_\ell$-sheaves on the local Hecke stack Hec_G, which are universally locally acyclic and have pullbacks to the affine Grassmannian being perverse relative to $\mathring{X}$. (For the moment, we assume $I = \{1\}$ and omit it from the notations Hec_G^I and $\mathscr{G}^I$.)

The category $\mathrm{Sat}_{G,\mathscr{G}}$ admits a natural tensor structure, coming from the fusion product. We modify the commutativity constraint in the usual manner (having to do with $2\check{\rho}$) to obtain a new tensor category ${}^+\mathrm{Sat}_{G,\mathscr{G}}$. This modification ensures that the normalized constant term functor, defined using a half-integral Tate twist $\overline{\mathbb{Q}}_\ell(\frac{1}{2})$, is symmetric monoidal.

Theorem B. *For a fixed $\overline{\mathbb{Q}}_\ell(\frac{1}{2})$, there is a canonical equivalence of tensor categories:*

$${}^+\mathrm{Sat}_{G,\mathscr{G}} \cong \mathrm{Rep}_{H,F_\vartheta}. \tag{0.9}$$

0.2.8. Theorem B is the special case of Theorem 2.4.4 for $I = \{1\}$, although the additional challenges presented by general I are mostly notational.

Assuming the general form of equivalence (0.9), we obtain a system of (non-symmetric monoidal) functors parametrized by nonempty finite sets I:

$$\begin{array}{ccccc} \mathrm{Rep}^{\mathrm{alg}}(({}^LH_{\mathring{X},\vartheta})^I) & \cong & \mathrm{Rep}_{H^I,{}^0F_\vartheta^I} & & \\ & & \big\downarrow{\scriptstyle \text{monoidal}} & & \\ & & \mathrm{Rep}_{H^I,F_\vartheta^I} & \cong & {}^+\mathrm{Sat}_{G,\mathscr{G}^I} \xrightarrow{\text{Shtuka}} \mathrm{Ind}(\mathrm{Lis}(\mathring{X}^I)). \end{array} \tag{0.10}$$

Here, the source is the category of continuous finite-dimensional representations of the product $({}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}},\vartheta})^{\mathrm{I}}$ whose restrictions to $\mathrm{H}_{\bar{\eta}}(\overline{\mathbb{Q}}_\ell)^{\mathrm{I}}$ lift to algebraic representations of $\mathrm{H}_{\bar{\eta}}^{\mathrm{I}}$. The vertical functor is the monoidal equivalence induced from the identification ${}^0\mathrm{F}_\vartheta \cong \mathrm{F}_\vartheta$ as $\mathbb{E}_1$-*monoidal* morphisms. The next two functors are the geometric Satake equivalence (0.9), respectively the cohomology of Shtukas discussed in §0.2.3.

There is furthermore an iterated variant of the functors (0.10), attached to ordered partitions of I. It is needed to equip the target of (0.10) with equivariance structures with respect to the partial Frobenii on $\mathring{\mathrm{X}}^{\mathrm{I}}$.

Given this input, we are in a position to apply the machinery of [Laf18, §5-7] and [Xue20b]: it proves that the image of (0.10) is contained in $\mathrm{Ind}(\mathrm{Rep}(\pi_1(\mathring{\mathrm{X}},\bar{\eta})^{\mathrm{I}}))$. The spectral decomposition of genuine cusp forms then follows verbatim from [Laf18, §9-11].

0.2.9. Finally, we mention one place in our proof of Theorem B where a new ingredient is needed. The assertion is of étale local nature on $\mathring{\mathrm{X}}$, so we may assume that G splits and replace $\mathring{\mathrm{X}}$ by any smooth curve X, not necessarily connected.

One unique feature of the metaplectic context is the absence of a natural fiber functor out of ${}^+\mathrm{Sat}_{\mathrm{G},\mathscr{G}}$, even at a geometric point of X. This creates difficulties in applying the Tannakian formalism.

Let us work with a fixed Borel subgroup and a maximal torus $\mathrm{T} \subset \mathrm{B} \subset \mathrm{G}$. The étale metaplectic cover μ induces one for T and defines an A-gerbe $\mathscr{G}_{\mathrm{T}}$ over the Hecke stack $\mathrm{Hec}_{\mathrm{T}}$. We thus have a constant term functor at our disposal:

$$\mathrm{CT}_{\mathrm{B}}(\check{\rho})[2\check{\rho}] : {}^+\mathrm{Sat}_{\mathrm{G},\mathscr{G}} \to \mathrm{Sat}_{\mathrm{T},\mathscr{G}_{\mathrm{T}}}. \tag{0.11}$$

Let us assume that the case for tori is already proved, so $\mathrm{Sat}_{\mathrm{T},\mathscr{G}_{\mathrm{T}}}$ is identified with the tensor category $\mathrm{Rep}_{\mathrm{T}_{\mathrm{H}},\mathrm{F}_\vartheta}$ of F_ϑ-twisted category of representations of the metaplectic dual torus T_{H} on lisse $\overline{\mathbb{Q}}_\ell$-sheaves. (The F_ϑ-twist invokes a natural surjection of the character sheaf of T_{H} onto $\hat{\mathrm{Z}}_{\mathrm{H}}$.)

The tensor category $\mathrm{Rep}_{\mathrm{T}_{\mathrm{H}},\mathrm{F}_\vartheta}$ does not admit a natural fiber functor to Lis(X), unless we undo the F_ϑ-twist. Therefore, we wish to twist both the source and target of (0.11) by $\mathrm{F}_\vartheta^{\otimes -1}$ and apply a relative Tannakian formalism to the resulting functor:

$$({}^+\mathrm{Sat}_{\mathrm{G},\mathscr{G}})_{\mathrm{F}_\vartheta^{\otimes -1}} \to (\mathrm{Sat}_{\mathrm{T},\mathscr{G}_{\mathrm{T}}})_{\mathrm{F}_\vartheta^{\otimes -1}} \cong \mathrm{Rep}_{\mathrm{T}_{\mathrm{H}}} \to \mathrm{Lis}(\mathrm{X}).$$

However, in order to twist ${}^+\mathrm{Sat}_{\mathrm{G},\mathscr{G}}$, we must construct a $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading on it compatible with the tensor structure. In the non-metaplectic context, this would be the $\pi_1(\mathrm{G})$-grading coming from the connected components of $\mathrm{Hec}_{\mathrm{G}}$. The $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading is in general *finer*. Its existence on the level of abelian categories poses no difficulty, but its compatibility with the tensor structure is not obvious.

We shall reduce the problem to studying the weights occurring in (0.11), which have to do with the behavior of the A-gerbe $\mathscr{G}$ on Mirkovic–Vilonen cycles. The new ingredient is a description of how $\mathscr{G}$ interacts with the action of the adjoint torus, which eventually reduces to a calculation of Deligne [Del96, §4] (as reformulated in [Zha22, §5.5].)

Our proof of Theorem B owes much intellectual debt to pioneering works on the subject by Finkelberg–Lysenko [FL10] and Reich [Rei12], although it does not rely on their results.[1] We have also benefitted from the notion of relative perversity, recently developed by Hansen and Scholze [HS21], which streamlined many arguments.

[1] The aforementioned challenge was already present in [FL10], but went unnoticed because of a mistake in [FL10, §4.2], where the "fiber functor" used in the Tannakian formalism was not symmetric monoidal (as pointed out in [Rei12, V.1]). The issue was not resolved in [Rei12], because the proof of its Lemma IV.7.8 mistakenly asserted that the $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading on ${}^+\mathrm{Sat}_{\mathrm{G},\mathscr{G}}$ could be obtained from the $\pi_1(\mathrm{G})$-grading.

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1. Preparation

This section collects some preliminary notions which will be used in the remainder of the article. We also use it as an opportunity to introduce notations.

The first topic is the formalism of ℓ-adic sheaves twisted by a gerbe banded by the group of units of the coefficient field. In particular, we explain in §1.4 how they encode genuine functions. The second topic is the notion of étale metaplectic covers and their L-groups, which we recall in §1.5-1.6.

1.1. A-gerbes.

1.1.1. We work over a base scheme S.

1.1.2. Suppose that G is an étale sheaf of groups over S. We denote by $\mathrm{B_S(G)}$ the classifying stack of G sheafified in the étale topology. The term G-*torsor* over an S-scheme X refers to a section of $\mathrm{B_S(G)}$ over X

If the base scheme S is clear from the context, we shall suppress it from the notation.

1.1.3. When G is furthermore abelian, we write $\mathrm{B}^n(\mathrm{G})$ for the n-fold delooping as an étale stack of ∞-groupoids for any integer $n \geq 1$, see [Zha22, §1].

For an S-scheme X, the ∞-groupoid of sections $\mathrm{Maps(X, B^{\mathit{n}}(G))}$ has homotopy groups described by the étale cohomology groups of X valued in G:

$$\pi_i \mathrm{Maps}(\mathrm{X}, \mathrm{B}^n(\mathrm{G})) \cong \mathrm{H}^{n-i}(\mathrm{X}, \mathrm{G}).$$

The term G-*gerbe* over an S-scheme X refers to a section of $\mathrm{B^2(G)}$ over X.

The 2-groupoid of G-gerbes over X carries a natural $\mathbb{E}_\infty$-monoidal structure, and we use $\otimes$ to denote the product operation.

Remark 1.1.4. This notion of G-gerbe is equivalent to the more classical notion of a "gerbe banded by G", which is a stack over X equipped with additional structures.

The dictionary goes as follows: given a section of $\mathrm{B^2(G)}$ over X, we let $\mathscr{G}$ be the étale stack over X whose sections over an X-scheme X_1 are *rigidifications* of the composition: $\mathrm{X}_1 \to \mathrm{X} \to \mathrm{B}^2(\mathrm{G})$, i.e. factorizations of it through the canonical section $e : \mathrm{S} \to \mathrm{B}^2(\mathrm{G})$.

In particular, the groupoid $\mathscr{G}(\mathrm{X}_1)$ is equipped with an action of the monoidal groupoid of G-torsors on X_1: each G-torsor t defines an automorphism of the canonical section $\mathrm{X}_1 \to \mathrm{S} \xrightarrow{e} \mathrm{B}^2(\mathrm{G})$ and carries $g \in \mathscr{G}(\mathrm{X}_1)$ to the composition $g \cdot t \in \mathscr{G}(\mathrm{X}_1)$.

1.1.5. Suppose that X is a connected S-scheme equipped with a geometric point $\bar{x}$. Let $\pi_1(\mathrm{X}, \bar{x})$ denote the profinite fundamental group.

Let Z be a locally constant étale sheaf of finite abelian groups over X. Its geometric fiber $\mathrm{Z}_{\bar{x}}$ is thus equipped with a continuous action of $\pi_1(\mathrm{X}, \bar{x})$.

Denote by $\mathscr{X}^2(\pi_1(\mathrm{X}, \bar{x}), \mathrm{Z}_{\bar{x}})$ the groupoid of short exact sequences of profinite groups:

$$1 \to \mathrm{Z}_{\bar{x}} \to \Pi \to \pi_1(\mathrm{X}, \bar{x}) \to 1,$$

such that the conjugation action of Π on $\mathrm{Z}_{\bar{x}}$ factors through the natural $\pi_1(\mathrm{X}, \bar{x})$-action.

1.1.6. To interpret Z-gerbes in terms of $\pi_1(X, \bar{x})$, the following condition on X is needed:

$$\text{For any finite abelian group A,} \\ \text{any class of } H^2(X, A) \text{ vanishes over a } \textit{finite} \text{ étale cover of X.} \tag{1.1}$$

Examples of such X include spectra of fields (automatic) and Henselian local rings ([Sta18, 09ZI]) as well as all connected affine $\mathbb{F}_p$-schemes ([Ach17, Theorem 1.1.1]), although we are only interested in the case where X is an affine curve over a finite field, where (1.1) can be verified directly.

An example of an S-scheme failing condition (1.1) is the projective line.

1.1.7. Let X, $\bar{x}$, Z be as in §1.1.5. Denote by $\mathrm{Maps}_{\bar{x}}(X, B^2(Z))$ the (1-)groupoid of A-gerbes over X rigidified along $\bar{x}$.

If X satisfies condition (1.1), the usual comparison between étale and Galois cohomology lifts to a canonical equivalence of Picard groupoids:

$$\mathrm{Maps}_{\bar{x}}(X, B^2(Z)) \cong \mathscr{Z}^2(\pi_1(X, \bar{x}), Z_{\bar{x}}). \tag{1.2}$$

Following [Wei18, §19], we refer to the image of a rigidified A-gerbe $(\mathscr{G}, \bar{g})$ under (1.2) as the "fundamental group" of $(\mathscr{G}, \bar{g})$ and denote it as follows:

$$1 \to Z_{\bar{x}} \to \pi_1(\mathscr{G}, \bar{g}) \to \pi_1(X, \bar{x}) \to 1. \tag{1.3}$$

Here, the rigidification $\bar{g}$ of $\mathscr{G}$ along $\bar{x}$ may equivalently be viewed as a geometric point of $\mathscr{G}$ lifting $\bar{x}$.

Construction of (1.2). Let $\widetilde{X}$ denote the universal cover of $(X, \bar{x})$: it is the pro-object of the category of finite étale X-schemes which co-represents the fiber functor $(p : X_1 \to X) \mapsto p^{-1}(\bar{x})$. In particular, $\bar{x}$ canonically lifts to $\widetilde{X}$, so we may view it as a geometric point of $\widetilde{X}$.

Let $\mathscr{G}$ be a Z-gerbe over X. Condition (1.1) implies that the pullback $\widetilde{\mathscr{G}}$ of $\mathscr{G}$ to $\widetilde{X}$ is constant. Thus for each $\sigma \in \pi_1(X, \bar{x})$, we obtain an automorphism:

$$\widetilde{\mathscr{G}}_{\bar{x}} \cong \widetilde{\mathscr{G}}_{\sigma(\bar{x})} \cong \widetilde{\mathscr{G}}_{\bar{x}}, \tag{1.4}$$

where the first map is provided by the constancy of $\widetilde{\mathscr{G}}$ and the second map is the descent datum of $\widetilde{\mathscr{G}}$ along $\widetilde{X} \to X$.

Any rigidification $\bar{g}$ of $\mathscr{G}$ along $\bar{x}$ may be viewed as a section of $\widetilde{\mathscr{G}}_{\bar{x}}$. The automorphism (1.4) acts on $\bar{g}$ as multiplication by a (set-theoretic) $Z_{\bar{x}}$-torsor t_σ.

The association $\sigma \mapsto t_\sigma$ is multiplicative in the following sense: for two elements $\sigma_1, \sigma_2 \in \pi_1(X, \bar{x})$, there is an identification of $Z_{\bar{x}}$-torsors:

$$(\sigma_2)^*(t_{\sigma_1}) \otimes t_{\sigma_2} \cong t_{\sigma_1 \sigma_2}, \tag{1.5}$$

where $(\sigma_2)^*(t_{\sigma_1})$ is the $Z_{\bar{x}}$-torsor induced from t_{σ_1} along σ_2^{-1} (*i.e.* $z \in Z_{\bar{x}}$ acts through σ_2). The identification (1.5) satisfies the natural cocycle condition. Furthermore, t_e is canonically trivialized, satisfying the unit condition with respect to (1.5).

Therefore, the union:

$$\pi_1(\mathscr{G}, \bar{g}) := \bigsqcup_{\sigma \in \pi_1(X, \bar{x})} t_\sigma.$$

defines an extension of $\pi_1(X, \bar{x})$ by $Z_{\bar{x}}$, with multiplication induced from (1.5). This concludes the definition of the functor (1.2) in the forward direction. It is symmetric monoidal respect to the natural symmetric monoidal structures on both sides.

To show that (1.2) is an equivalence, we observe that it induces the isomorphism between $H^2(X, Z)$ and $H^2(\pi_1(X, \bar{x}), Z_{\bar{x}})$ on π_0 (oweing to condition (1.1)). On π_1, it induces the isomorphism of abelian groups between $\mathrm{Maps}_{\bar{x}}(X, B(Z))$ and maps $f : \pi_1(X, \bar{x}) \to Z_{\bar{x}}$ satisfying $\sigma_2^{-1}(f(\sigma_1))f(\sigma_2) = f(\sigma_1\sigma_2)$ for each $\sigma_1, \sigma_2 \in \pi_1(X, \bar{x})$. □

Remark 1.1.8. If $Z = \underline{A}$ is the constant étale sheaf with values in a finite abelian group A, the isomorphism $\underline{A}_{\bar{x}} \cong A$ induces a retract:

$$\mathrm{Maps}(X, B^2(\underline{A})) \to \mathrm{Maps}_{\bar{x}}(X, B^2(\underline{A})), \quad \mathscr{G} \mapsto \mathscr{G} \otimes \mathscr{G}_{\bar{x}}^{\otimes -1}. \tag{1.6}$$

On the other hand, the groupoid $\mathscr{Z}^2(\pi_1(X, \bar{x}), \underline{A}_{\bar{x}})$ is equivalent to that of central extensions of $\pi_1(X, \bar{x})$ by A, which we denote by $\mathrm{CExt}(\pi_1(X, \bar{x}), A)$. Hence, the composition of (1.6) with (1.2) is a functor of Picard groupoids:

$$\mathrm{Maps}(X, B^2(\underline{A})) \to \mathrm{CExt}(\pi_1(X, \bar{x}), A). \tag{1.7}$$

The functor (1.7) induces an equivalence after 1-truncation.

1.2. Twisted ℓ-adic sheaves.

1.2.1. We continue to work over a base scheme S. We fix a prime ℓ invertible on S and an algebraic closure $\mathbb{Q}_\ell \subset \overline{\mathbb{Q}}_\ell$.

Let E be an intermediate field $\mathbb{Q}_\ell \subset E \subset \overline{\mathbb{Q}}_\ell$ and $A \subset E^\times$ be a finite subgroup whose order is also invertible on S.

1.2.2. For any S-scheme X, we have the ∞-category $\mathrm{Shv}(X, E)$ of constructible complexes of E-sheaves on X. This is the ∞-category denoted by $D_{\mathrm{cons}}(X, E)$ in [HRS20] where E is equipped with the ℓ-adic topology.

1.2.3. In the presence of an A-gerbe $\mathscr{G}$ on X, there is a variant: the ∞-category $\mathrm{Shv}_{\mathscr{G}}(X)$ of *$\mathscr{G}$-twisted constructible complexes of* E-*sheaves* on X.

To define it, consider the abelian category $\mathscr{A}(X)$ of proétale E-sheaves on X. The association $X_1 \mapsto \mathscr{A}(X_1)$ is a stack of abelian categories on the small étale site of X, to be denoted (temporarily) by $\underline{\mathscr{A}}$.

Since A is a subgroup of $E^\times$, it acts on the identity endofunctor of $\underline{\mathscr{A}}$. The construction of [Zha22, Appendix A] then yields a stack $\underline{\mathscr{A}}_{\mathscr{G}}$. Its global section is an abelian category $\mathscr{A}_{\mathscr{G}}(X)$. Then we may form its derived ∞-category $D_{\mathscr{G}}(X)$ and $\mathrm{Shv}_{\mathscr{G}}(X) \subset D_{\mathscr{G}}(X)$ is the full ∞-subcategory characterized by the constructibility condition of [HRS20, Definition 1.1].

1.2.4. For brevity, we shall call an object of $\mathrm{Shv}_{\mathscr{G}}(X)$ a "$\mathscr{G}$-twisted E-sheaf" on X.

Such an object is said to be *lisse* or a *$\mathscr{G}$-twisted* E-*local system* if it is invertible and belongs to the heart $\mathscr{A}_{\mathscr{G}}(X)$ of $D_{\mathscr{G}}(X)$. They form an abelian category $\mathrm{Lis}_{\mathscr{G}}(X)$.

Remark 1.2.5. Any trivialization of $\mathscr{G}$ induces an equivalence of ∞-categories between $\mathrm{Shv}_{\mathscr{G}}(X)$ and $\mathrm{Shv}(X)$.

Since $\mathscr{G}$ is locally trivial in the étale topology, constructions on $\mathrm{Shv}(X)$ of étale local nature automatically carry over to $\mathrm{Shv}_{\mathscr{G}}(X)$.

1.2.6. It is convenient to consider the 2-category of pairs $(X, \mathscr{G})$, where X is an S-scheme (or more generally, an S-prestack) and $\mathscr{G}$ is an A-gerbe over X.

A morphism $(X_1, \mathscr{G}_1) \to (X_2, \mathscr{G}_2)$ consists of a morphism $f : X_1 \to X_2$ and an isomorphism $\alpha : \mathscr{G}_1 \xrightarrow{\sim} f^*(\mathscr{G}_2)$.

A 2-morphism $(f_1, \alpha_1) \to (f_2, \alpha_2)$ is an equality $f_1 = f_2 : X_1 \to X_2$ together with a 2-morphism $\alpha_1 \to \alpha_2$ in the 2-category of A-gerbes on X_1. All 2-morphisms are invertible.

1.2.7. Given a morphism $(X_1, \mathscr{G}_1) \to (X_2, \mathscr{G}_2)$ as in §1.2.6, there is a pullback functor $f^* : \mathrm{Shv}_{\mathscr{G}_2}(X_2) \to \mathrm{Shv}_{\mathscr{G}_1}(X_1)$.

If f is separated and of finite presentation, we also have a functor $f_! : \mathrm{Shv}_{\mathscr{G}_1}(X_1) \to \mathrm{Shv}_{\mathscr{G}_2}(X_2)$. (By our convention, functors are derived unless otherwise stated.)

The adjunctions (f^*, f_*), $(f^!, f_!)$ exist on the twisted categories of E-sheaves whenever they exist on the untwisted ones.

Indeed, these functors are constructed from the usual functors by étale descent, in view of Remark 1.2.5.

Remark 1.2.8. Suppose that X is connected, Noetherian, and geometrically unibranch. Let $\bar{x}$ be a geometric point of X. Recall that lisse E-sheaves on X are equivalent to continuous $\pi_1(\mathrm{X}, \bar{x})$-representations on finite-dimensional E-vector spaces, the functor being taking fibers at $\bar{x}$.

Assume that X satisfies condition (1.1). Then an A-gerbe $\mathscr{G}$ on X with rigidification $\bar{g}$ along $\bar{x}$ defines a central extension (1.3).

Taking fibers at $\bar{x}$ yields an equivalence between $\mathscr{G}$-twisted lisse E-sheaves on X and continuous $\pi_1(\mathscr{G}, \bar{g})$-representations on finite-dimensional E-vector spaces such that A acts through the inclusion $\mathrm{A} \subset \mathrm{E}^\times$.

1.3. A vanishing lemma.

1.3.1. We remain in the context of §1.2.1.

1.3.2. Let G be an étale sheaf of groups over S. There is an equivalence of groupoids between A-gerbes on B(G) rigidified along $e : \mathrm{S} \to \mathrm{B(G)}$ (simply called "rigidified A-gerbes" below) and monoidal morphisms $\mathrm{G} \to \mathrm{B}(\underline{\mathrm{A}})$.

In turn, monoidal morphisms $\mathrm{G} \to \mathrm{B}(\underline{\mathrm{A}})$ are equivalent to multiplicative A-torsors on G. They induce character E-local systems on G along the inclusion $\mathrm{A} \subset \mathrm{E}^\times$.

1.3.3. Suppose that G acts on an S-scheme X. Let $\mathscr{G}$ be a rigidified A-gerbe on B(G). Denote by $\chi_{\mathscr{G}}$ the induced character E-local system on G.

By abuse of notation, we write $\mathrm{Shv}_{\mathscr{G}}(\mathrm{X/G})$ for the ∞-category of E-sheaves on X/G twisted by the pullback of $\mathscr{G}$ along $\mathrm{X/G} \to \mathrm{B(G)}$.

The ∞-category $\mathrm{Shv}_{\mathscr{G}}(\mathrm{X/G})$ admits a more concrete description: it is the ∞-category of E-sheaves on X equipped with G-equivariance structures against the character E-local system $\chi_{\mathscr{G}}$.

1.3.4. For an S-scheme X, we write $\Gamma(\mathrm{X}, -)$ (resp. $\Gamma_c(\mathrm{X}, -)$) for the direct image functors $p_*(-)$ (resp. $p_!(-)$) along the structure morphism $p : \mathrm{X} \to \mathrm{S}$. Similar conventions apply to $\mathrm{H}^i(\mathrm{X}, -)$ and $\mathrm{H}^i_c(\mathrm{X}, -)$.

The following lemma is a variant of the fact that an E-sheaf equivariant against a non-trivial character on the stabilizer group must vanish.

Lemma 1.3.5. *Let $\mathscr{G}$ be a rigidified* A*-gerbe on* B(G)*, with induced character* E*-local system* $\chi_{\mathscr{G}}$ *on* G. *If* $\mathrm{H}^0(\mathrm{G}, \chi_{\mathscr{G}}) = 0$*, then any object* $\mathscr{F} \in \mathrm{Shv}_{\mathscr{G}}(\mathrm{X/G})$ *satisfies:*

$$\mathrm{H}^i_c(\mathrm{X}, \mathscr{F}) = 0 \quad \text{for all } i \geq 0.$$

(In particular, we find $\mathrm{Shv}_{\mathscr{G}}(\mathrm{B(G)}) = 0$ *by setting* $\mathrm{X} = \mathrm{S}$.*)*

Proof. Write $f : \mathrm{X} \to \mathrm{X/G}$ for the quotient in the étale topology. It suffices to show $f_!(\mathscr{F}) = 0$.

Consider the Cartesian diagram:

$$\begin{array}{ccc} \mathrm{X} \times \mathrm{G} & \xrightarrow{\mathrm{act}} & \mathrm{X} \\ \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow{\scriptstyle f} \\ \mathrm{X} & \xrightarrow{f} & \mathrm{X/G} \end{array} \tag{1.8}$$

where act (resp. pr) stands for the action (resp. projection) map.

We view $\mathscr{F}$ as an E-sheaf on X which is G-equivariant against $\chi_{\mathscr{G}}$. Base change along (1.8), the equivariance structure, and projection formula imply an isomorphism:

$$f^* f_!(\mathscr{F}) \cong \mathscr{F} \boxtimes \Gamma_c(\mathrm{G}, \chi_{\mathscr{G}}). \tag{1.9}$$

It remains to show $\Gamma_c(\mathrm{G}, \chi_{\mathscr{G}}) = 0$.

For $\mathrm{X} = \mathrm{G}$ and $\mathscr{F} = \chi_{\mathscr{G}}$, (1.9) reads:

$$\mathrm{E} \boxtimes \Gamma_c(\mathrm{G}, \chi_{\mathscr{G}}) \cong \chi_{\mathscr{G}} \boxtimes \Gamma_c(\mathrm{G}, \chi_{\mathscr{G}}). \tag{1.10}$$

In particular, any nonzero $\mathrm{H}^i_c(\mathrm{G}, \chi_{\mathscr{G}})$ implies the existence of a nonzero section of $\chi_{\mathscr{G}}$. □

1.3.6. Let us discuss an example for which the condition of Lemma 1.3.5 is satisfied.

Suppose that $\mathrm{G} = \mathbb{G}_{m,\mathrm{S}}$ is the multiplicative group over S. For each integer $n \geq 0$ invertible over S, the degree-n Kummer cover of $\mathbb{G}_{m,\mathrm{S}}$ defines a character μ_n-torsor on $\mathbb{G}_{m,\mathrm{S}}$.

In particular, we have a monoidal morphism:

$$\Psi : \mathbb{G}_{m,\mathrm{S}} \to \lim_{\substack{n\geq 0 \\ \text{invertible}}} \mathrm{B}(\mu_n). \tag{1.11}$$

For each section a of the étale sheaf $\underline{\mathrm{A}}(-1)$, we thus obtain a monoidal morphism $a_*(\Psi) : \mathbb{G}_{m,\mathrm{S}} \to \mathrm{B}(\underline{\mathrm{A}})$. We also write $\Psi^a := a_*(\Psi)$ and keep the same notation for the induced character E-local system on $\mathbb{G}_{m,\mathrm{S}}$.

When a is nowhere vanishing on S, there holds $\mathrm{H}^0(\mathbb{G}_{m,\mathrm{S}}, \Psi^a) = 0$. Indeed, this can be checked at geometric points $\bar{s} \in \mathrm{S}$, where it follows from the vanishing of $\pi_1(\mathbb{G}_{m,\bar{s}}, e)$-invariants of the corresponding 1-dimensional character.

1.4. Frobenius.

1.4.1. Suppose that the base scheme $\mathrm{S} = \mathrm{Spec}(k)$, where k is a finite field of cardinality q. Let A be a finite abelian group whose order is coprime to q.

For any k-scheme X, we write $\mathrm{Fr}_{\mathrm{X}} : \mathrm{X} \to \mathrm{X}$ for the absolute Frobenius endomorphism: it acts as identity on the topological space $|\mathrm{X}|$ and the qth power map on $\mathscr{O}_{\mathrm{X}}$.

1.4.2. Consider the special case $x = \mathrm{Spec}(k)$. Then any A-torsor t on x defines a character $\mathrm{Gal}(\bar{k}/k) \to \mathrm{A}$ for any algebraic closure $k \subset \bar{k}$, and the image of the geometric Frobenius $\varphi_x \in \mathrm{Gal}(\bar{k}/k)$ may be called the *trace-of-Frobenius* of t.

When a coefficient field E with $\mathrm{A} \subset \mathrm{E}^\times$ is supplied, this is indeed the trace of φ_x on the 1-dimensional representation induced from t along $\mathrm{A} \subset \mathrm{E}^\times$.

1.4.3. We shall describe an analogous construction for A-gerbes on a k-scheme (or k-stack) X. It is helpful to perform this construction in two steps:

$$\begin{array}{ccccc} \mathrm{X} & & \mathrm{X}^{\mathrm{Fr}} & & \mathrm{X}(k) \\ \downarrow{\scriptstyle \mathscr{G}} & \Rightarrow & \downarrow{\scriptstyle \mathrm{Tr}(\mathrm{Fr}|\mathscr{G})} & \Rightarrow & \downarrow{\scriptstyle \mathrm{Tr}(\mathrm{Fr}|\mathscr{G})(k)} \\ \mathrm{B}^2(\underline{\mathrm{A}}) & & \mathrm{B}(\underline{\mathrm{A}}) & & \mathrm{B}(\mathrm{A}) \end{array} \tag{1.12}$$

In other words, we shall first extract an étale A-torsor $\mathrm{Tr}(\mathrm{Fr} \mid \mathscr{G})$ over the Fr_{X}-fixed point locus $\mathrm{X}^{\mathrm{Fr}} \subset \mathrm{X}$, defined to be the fiber product:

$$\begin{array}{ccc} \mathrm{X}^{\mathrm{Fr}} & \longrightarrow & \mathrm{X} \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{id}, \mathrm{Fr}_{\mathrm{X}})} \\ \mathrm{X} & \xrightarrow{\Delta} & \mathrm{X} \times \mathrm{X} \end{array} \tag{1.13}$$

and then set $\mathrm{Tr}(\mathrm{Fr} \mid \mathscr{G})(k)$ to be its set (or groupoid) of k-points.

1.4.4. Recall that for any k-scheme X, the endofunctor $\mathrm{Fr}_{\mathrm{X}}^*$ on the 2-groupoid of A-gerbes over X is naturally isomorphic to the identity (the "baffling theorem" [Sta18, 03SN]).

Let us explicitly describe the value of this natural isomorphism at an A-gerbe $\mathscr{G}$:

$$\mathrm{Fr}_{\mathrm{X}}^*(\mathscr{G}) \cong \mathscr{G}. \tag{1.14}$$

For any étale morphism $f : \mathrm{X}_1 \to \mathrm{X}$, the groupoid $\mathrm{Fr}_{\mathrm{X}}^*(\mathscr{G})(\mathrm{X}_1)$ is the filtered colimit of $\mathscr{G}(\mathrm{U})$ over étale morphisms $u : \mathrm{U} \to \mathrm{X}$ through which $\mathrm{Fr}_{\mathrm{X}} \circ f$ factors. This index category has an initial object, given by $(\mathrm{U}, u) = (\mathrm{X}_1, f)$ and the factorization $\mathrm{Fr}_{\mathrm{X}} \circ f = f \circ \mathrm{Fr}_{\mathrm{X}_1}$. The colimit is thus identified with $\mathscr{G}(\mathrm{X}_1)$.

Let us now give two constructions of $\mathrm{Tr}(\mathrm{Fr} \,|\, \mathscr{G})$.

Construction 1. Since Fr_{X} restricts to the identity map on X^{Fr}, we obtain a "tautological" identification $\mathrm{Fr}_{\mathrm{X}}^*(\mathscr{G}) \cong \mathscr{G}$ of A-gerbes over X^{Fr}.

The A-torsor $\mathrm{Tr}(\mathrm{Fr}_{\mathrm{X}} \,|\, \mathscr{G})$ over X^{Fr} is defined so that the action by it renders the following diagram of A-gerbes over X^{Fr} commutative:

$$\begin{array}{ccc} \mathrm{Fr}_{\mathrm{X}}^*(\mathscr{G}) & \xrightarrow{\mathrm{taut}} & \mathscr{G} \\ \Big\downarrow \mathrm{id} & & \Big\downarrow \cdot \mathrm{Tr}(\mathrm{Fr}_{\mathrm{X}}|\mathscr{G}) \\ \mathrm{Fr}_{\mathrm{X}}^*(\mathscr{G}) & \xrightarrow{(1.14)} & \mathscr{G} \end{array} \tag{1.15}$$

where taut refers to the tautological identification. □

Remark 1.4.5. For an étale morphism $f : \mathrm{X}_1 \to \mathrm{X}^{\mathrm{Fr}}$, the automorphism of $\mathscr{G}(\mathrm{X}_1)$ defined by the action of $\mathrm{Tr}(\mathrm{Fr}_{\mathrm{X}} \,|\, \mathscr{G})$ is the pullback along the X^{Fr}-automorphism $\mathrm{Fr}_{\mathrm{X}_1}^{-1} : \mathrm{X}_1 \to \mathrm{X}_1$. (Note that Fr_{X} being invertible over X^{Fr} implies that $\mathrm{Fr}_{\mathrm{X}_1}$ is as well.)

Construction 2. The identification (1.14) yields an isomorphism between the endomorphism $\mathrm{Fr}_{\mathrm{B}^2(\underline{\mathrm{A}})}$ of $\mathrm{B}^2(\underline{\mathrm{A}})$ and the identity.

Hence we find an isomorphism:

$$\mathrm{B}^2(\underline{\mathrm{A}})^{\mathrm{Fr}} \cong \mathrm{B}^2(\underline{\mathrm{A}}) \times \mathrm{B}(\underline{\mathrm{A}}), \quad (\mathscr{G}, \alpha) \mapsto (\mathscr{G}, \mathscr{G}^{-1} \otimes \alpha), \tag{1.16}$$

where the isomorphism $\alpha : \mathscr{G} \cong \mathrm{Fr}_{\mathrm{X}}^*(\mathscr{G})$ is viewed as an automorphism of $\mathscr{G}$ via the identification (1.14), so $\mathscr{G}^{-1} \otimes \alpha$ is an automorphism of the trivial A-gerbe, *i.e.* an A-torsor.

The A-torsor $\mathrm{Tr}(\mathrm{Fr}_{\mathrm{X}} \,|\, \mathscr{G})$ over X^{Fr} is set to be the composition:

$$\mathrm{X}^{\mathrm{Fr}} \to \mathrm{B}^2(\underline{\mathrm{A}})^{\mathrm{Fr}} \to \mathrm{B}(\underline{\mathrm{A}}),$$

where the second map is the projection of (1.16) onto its second factor. □

1.4.6. Let us argue that $\mathrm{Tr}(\mathrm{Fr} \,|\, \mathscr{G})(k)$ is indeed a set-theoretic A-torsor over $\mathrm{X}(k)$.

Namely, we must show that its fiber over any $x \in \mathrm{X}(k)$ is nonempty. To see this, it suffices to note that $\mathrm{H}^2(x, \mathrm{A}) = 0$, so any A-gerbe over x admits a section. The choice of any such section trivializes $\mathrm{Tr}(\mathrm{Fr} \,|\, \mathscr{G})$ over x.

Remark 1.4.7. Suppose now that $x = \mathrm{Spec}(k)$ and an algebraic closure $k \subset \bar{k}$ is chosen. We write $\bar{x} = \mathrm{Spec}(\bar{k})$.

Then the fiber of $\mathrm{Tr}(\mathrm{Fr} \,|\, \mathscr{G})(k)$ over $x \in \mathrm{X}(k)$ is identified with the preimage of $\pi_1(\mathscr{G}, \bar{x})$ at the geometric Frobenius element $\varphi_x \in \mathrm{Gal}(\bar{k}/k)$.

Indeed, since the fiber of $\mathrm{Tr}(\mathrm{Fr} \,|\, \mathscr{G})(k)$ over x is nonempty, it is identified with the fiber of $\mathrm{Tr}(\mathrm{Fr} \,|\, \mathscr{G})(\bar{k})$ over $\bar{x}$. Its description in Remark 1.4.5 coincides with the definition of t_σ in §1.1.7 for $\sigma = \varphi_x$ and any rigidification of $\mathscr{G}$ along $\bar{x}$.

Remark 1.4.8. Suppose that $k \subset k_1$ is a finite extension. Let X_1 be a k_1-scheme, whose restriction of scalars along $k \subset k_1$ is denoted by $X := \mathrm{res}(X_1)$. To each A-gerbe $\mathscr{G}_1$ over X_1, we may associate an A-gerbe $\mathrm{Nm}(\mathscr{G}_1)$ over X. To define it, we restrict $\mathscr{G}_1$ along the counit map $X_{k_1} := X \times \mathrm{Spec}(k_1) \to X_1$ and take its image under the norm map:

$$\Gamma(X_{k_1}, A[2]) \to \Gamma(X, A[2]),$$

which exists thanks to $X_{k_1} \to X$ being finite étale.

The trace-of-Frobenius construction may be performed for $\mathscr{G}_1$ (using the $|k_1|$th power Frobenius) as well as $\mathrm{Nm}(\mathscr{G}_1)$ (using the $|k|$th power Frobenius). They yield canonically isomorphic set-theoretic A-torsors:

$$\begin{array}{ccc} \mathrm{Tr}(\mathrm{Fr}_{X_1} \mid \mathscr{G}_1) & \cong & \mathrm{Tr}(\mathrm{Fr}_X \mid \mathrm{Nm}(\mathscr{G}_1)) \\ \downarrow & & \downarrow \\ X_1(k_1) & \cong & X(k) \end{array} \tag{1.17}$$

1.4.9. Let us bring in the coefficient field E as in §1.2.1 and assume that $A \subset E^\times$.

Let X be a k-scheme locally of finite type equipped with an A-gerbe $\mathscr{G}$. Denote by $\widetilde{X}$ the set-theoretic A-torsor $\mathrm{Tr}(\mathrm{Fr} \mid \mathscr{G})(k)$ over $X(k)$.

Let $\mathrm{Fun}_c(\widetilde{X}, A \subset E^\times)$ denote the E-vector space of compactly supported functions $f : \widetilde{X} \to E$ such that $f(\tilde{x} \cdot a) = f(\tilde{x}) \cdot a$ for each $a \in A$. Elements of $\mathrm{Fun}_c(\widetilde{X}, A \subset E^\times)$ are called *genuine functions* over $\widetilde{X}$ with respect to the inclusion $A \subset E^\times$.

In this set-up, there is a canonical isomorphism of E-vector spaces:

$$\mathrm{Fun}_c(\widetilde{X}, A \subset E^\times) \cong H^0_c(X^{\mathrm{Fr}}, \mathrm{Tr}(\mathrm{Fr}_X \mid \mathscr{G})^{\otimes -1}), \tag{1.18}$$

where $H^0_c(X^{\mathrm{Fr}}, -)$ denotes the colimit of functors $H^0_c(U, -)$ over quasi-compact open subschemes $U \subset X^{\mathrm{Fr}}$.

1.4.10. We are now in a position to explain how genuine functions arise from twisted E-local systems as defined in §1.2: this is the mechanism by which the cohomology of Shtukas will define genuine automorphic forms.

Consider the A-gerbe $\mathscr{G} \boxtimes (\mathscr{G}^{\otimes -1})$ over $X \times X$. In reference to (1.13), its restriction along Δ is canonically trivial, as is its restriction along $(\mathrm{id}, \mathrm{Fr}_X)$ by the isomorphism (1.14).

In particular, the restriction $\mathscr{G}_0$ of $\mathscr{G} \boxtimes (\mathscr{G}^{\otimes -1})$ to X^{Fr} admits two sections, corresponding to the two circuits of (1.13). If we write $g \in \mathscr{G}_0(X^{\mathrm{Fr}})$ for the section induced from the lower circuit, then the section induced from the upper is identified with $g \cdot \mathrm{Tr}(\mathrm{Fr}_X \mid \mathscr{G})$.

Now, we let $\mathscr{F}$ be the $\mathscr{G}_0$-twisted E-local system on X^{Fr}, which is identified with $\underline{E}$ using the section g (see Remark 1.2.5). The same $\mathscr{F}$ is identified with $\mathrm{Tr}(\mathrm{Fr}_X \mid \mathscr{G})^{\otimes -1}$ using the section $g \cdot \mathrm{Tr}(\mathrm{Fr}_X \mid \mathscr{G})$.

In other words, taking $H^0_c(X^{\mathrm{Fr}}, \mathscr{F})$ using the section of $\mathscr{G}_0$ induced from the *upper* circuit of (1.13) yields the E-vector space $\mathrm{Fun}_c(\widetilde{X}, A \subset E^\times)$.

1.5. Étale metaplectic covers.

1.5.1. Let S be a scheme and A be a finite abelian group whose order is invertible on S.

Suppose that X is an S-scheme and $G \to X$ is a smooth affine group scheme.

1.5.2. An *étale metaplectic cover* of $G \to X$ with values in A is defined to be a section of $B^4\underline{A}(1)$ over $B_X(G)$ equipped with a rigidification along $e : X \to B_X(G)$.

Recall that the subscript in B_X means taking the classifying stack of G relative to X (as opposed to the default base scheme S.)

Note that a rigidified section of $B^4\underline{A}(1)$ over $B_X(G)$ may be equivalently viewed as a morphism of $\mathbb{E}_1$-monoidal stacks $G \to B^3_X\underline{A}(1)$ over X.

Remark 1.5.3. This definition agrees with the one in [Zha22, §2] and we refer the reader to *op.cit.* for its relationship with classical metaplectic covers as well as their geometrization by means of K-theory.

It is imperative to point out that this definition is essentially contained in [Del96].

1.5.4. Suppose that X is the spectrum of a local nonarchimedean field F_x. Then an étale metaplectic cover μ defines a central extension of topological groups:[2]

$$1 \to A \to \widetilde{G}_x \to G(F_x) \to 1. \tag{1.19}$$

If X is instead the spectrum of the rings of integers $\mathscr{O}_x \subset F_x$, then the extension (1.19) produced by restricting (G, μ) to $\mathrm{Spec}(F_x)$ admits a canonical splitting over $G(\mathscr{O}_x)$.

1.5.5. If X is the spectrum of a global field F without real places, then an étale metaplectic cover μ defines a central extension of topological groups:

$$1 \to A \to \widetilde{G}_F \to G(\mathbb{A}_F) \to 1, \tag{1.20}$$

where $\mathbb{A}_F$ denotes the topological ring of adèles of F. Furthermore, (1.20) is equipped with a canonical splitting over G(F).

For each nonarchimedean place x of X, the restriction of (1.20) along the inclusion $G(F_x) \subset G(\mathbb{A}_F)$ recovers the central extension (1.19).

When A occurs as a subgroup of $E^\times$ for a field E, we have the E-vector space:

$$\mathrm{Fun}(\widetilde{G}_F/G(F), A \subset E^\times) \tag{1.21}$$

of G(F)-invariant locally constant functions $f : \widetilde{G}_F \to E$ satisfying $f(\tilde{x} \cdot a) = f(\tilde{x}) \cdot a$ for every $\tilde{x} \in \widetilde{G}_F$ and $a \in A$. They are called *genuine automorphic forms* on $\widetilde{G}_F$.

Roughly speaking, the Langlands program for étale metaplectic covers seeks to understand the decomposition of various subspaces of (1.21) according to "spectral data", defined in terms of an L-group associated to (G, μ).

1.6. The L-group.

1.6.1. We keep the notations of §1.5.1. Furthermore, we bring in the coefficient field E as in §1.2.1 and assume that $A \subset E^\times$.

1.6.2. We also assume that $G \to X$ is a reductive group scheme. Let Λ (resp. $\check{\Lambda}$) be the sheaf of cocharacters of the universal Cartan $T \to X$ of G.

The based root data of G consist of a sheaf of coroots (resp. simple coroots) Φ (resp. Δ) with $\Delta \subset \Phi \subset \Lambda$, a sheaf of roots (resp. simple roots) $\check{\Phi}$ (resp. $\check{\Delta}$) with $\check{\Delta} \subset \check{\Phi} \subset \check{\Lambda}$, and an isomorphism $\Phi \cong \check{\Phi}$. The image of $\alpha \in \Phi$ under this isomorphism is denoted by $\check{\alpha}$.

1.6.3. To each étale metaplectic cover μ of $G \to X$ with values in A, the recipe of [Zha22, §6] defines its *metaplectic dual data* (H, F), where:

(1) H is a locally constant étale sheaf over X of pinned split reductive groups over E;
(2) $F : \hat{Z}_H \to B^2_X(\underline{A})$ is an $\mathbb{E}_\infty$-monoidal morphism.

Here, Z_H denotes the center of H, $\hat{Z}_H$ the abelian group of its characters, viewed as a locally constant étale sheaf of abelian groups over X.

We shall partially recall the construction of (H, F) below.

[2]The construction of (1.19) uses local Tate duality, which requires fixing an isomorphism $\mathrm{Gal}(\bar{k}_x/k_x) \cong \hat{\mathbb{Z}}$, where k_x denotes the residue field of F_x. We normalize this isomorphism so that $1 \in \hat{\mathbb{Z}}$ corresponds to the *geometric* Frobenius element.

Remark 1.6.4. The construction of H is due to Lusztig [Lus93] and its role in the theory of metaplectic covers is explained by Finkelberg–Lysenko [FL10] and McNamara [McN12].

The construction of F is essentially due to Weissman [Wei18] when μ comes from K-theory and due to Gaitsgory–Lysenko [GL18] when X is a smooth curve (following *a priori* different approaches).

1.6.5. Given an étale metaplectic cover μ of $\mathrm{G} \to \mathrm{X}$, we first extract a triplet of invariants $(\mathrm{Q}, \mathrm{F}^\sharp, \varphi)$, where:

(1) Q is an $\underline{\mathrm{A}}(-1)$-valued quadratic form on Λ;
(2) $\mathrm{F}^\sharp : \Lambda^\sharp \to \mathrm{B}^2_{\mathrm{X}}(\underline{\mathrm{A}})$ is an $\mathbb{E}_\infty$-monoidal morphism, or equivalently an extension of stacks of Picard groupoids over X:
$$\mathrm{B}_{\mathrm{X}}(\underline{\mathrm{A}}) \to \widetilde{\Lambda}^\sharp \to \Lambda^\sharp,$$
(3) φ is an $\mathbb{E}_\infty$-monoidal trivialization of $\mathrm{F}^\sharp$ over $\Lambda^{\sharp,r} \subset \Lambda^\sharp$.

Here, $\Lambda^\sharp \subset \Lambda$ denotes the kernel of the symmetric form b associated to Q and $\Lambda^{\sharp,r} \subset \Lambda$ the $\mathbb{Z}$-span of the set:
$$\mathrm{ord}(\mathrm{Q}(\alpha)) \cdot \alpha, \quad \text{for each } \alpha \in \Phi.$$

The fact that $\Lambda^{\sharp,r}$ belongs to $\Lambda^\sharp$ follows from the equality satisfied by Q:
$$b(\alpha, \lambda) = \mathrm{Q}(\alpha)\langle\check{\alpha}, \lambda\rangle, \quad \text{for each } \alpha \in \Phi, \lambda \in \Lambda. \tag{1.22}$$

1.6.6. Let us sketch the construction of $(\mathrm{Q}, \mathrm{F}^\sharp, \varphi)$ and provide pointers to [Zha22]. The construction is performed étale locally on X using a Borel subgroup $\mathrm{B} \subset \mathrm{G}$, but it turns out to be independent of this choice ([Zha22, §5.2]) and thus globalizes.

First, we restrict μ along $\mathrm{B}(\mathrm{B}) \to \mathrm{B}(\mathrm{G})$, which canonically descends to an étale metaplectic cover μ_{T} of T.

The quadratic form Q is the unique discrete invariant of μ_{T}, in view of the isomorphism between $\mathrm{H}^4(\mathrm{BT}, \mathrm{A}(1))$ and quadratic forms on Λ ([Zha22, §4.3]).

Next, the restriction of μ_{T} to $\mathrm{B}(\mathrm{T}^\sharp)$ acquires a canonical $\mathbb{E}_\infty$-monoidal structure ([Zha22, §4.6]). Thus we obtain a $\mathbb{E}_\infty$-monoidal morphism $\mathrm{T}^\sharp \to \mathrm{B}^3_{\mathrm{X}}\underline{\mathrm{A}}(1)$, which yields $\mathrm{F}^\sharp$ by taking the associated sheaves of rigidified sections over $\mathbb{G}_{m,\mathrm{X}}$.

The trivialization φ arises from a calculation with SL_2 ([Zha22, §6.1.5]).

Remark 1.6.7. It is possible to enhance the data $(\mathrm{Q}, \mathrm{F}^\sharp, \varphi)$ to complete invariants of étale metaplectic covers ([Zha22, §5.1]), although we will not need them.

Note that due to the 2-categorical nature of these data, they are more difficult to state than their K-theoretic analogues defined in [BD01].

Construction of (H, F)*.* Let us write $\check{\Lambda}^\sharp$ for the $\mathbb{Z}$-linear dual of $\Lambda^\sharp$. For each $\alpha \in \Phi$, we set
$$\alpha^\sharp = \mathrm{ord}(\mathrm{Q}(\alpha)) \cdot \alpha \qquad \in \Lambda^\sharp;$$
$$\check{\alpha}^\sharp = \mathrm{ord}(\mathrm{Q}(\alpha))^{-1} \cdot \check{\alpha} \quad \in \check{\Lambda}^\sharp.$$

Let $\Phi^\sharp$ (resp. $\check{\Phi}^\sharp$) be the span of $\alpha^\sharp$ (resp. $\check{\alpha}^\sharp$) and $\Delta^\sharp$ (resp. $\check{\Delta}^\sharp$) its subset corresponding to $\alpha \in \Delta$ (resp. $\check{\alpha} \in \check{\Delta}$). Then the collection $\Delta^\sharp \subset \Phi^\sharp \subset \Lambda^\sharp$, $\check{\Delta}^\sharp \subset \check{\Phi}^\sharp \subset \check{\Lambda}^\sharp$, with bijection $\Phi^\sharp \cong \check{\Phi}^\sharp$, $\alpha^\sharp \mapsto \check{\alpha}^\sharp$, defines a locally constant sheaf of based root data over X.

The sheaf H is defined to be the associated sheaf of pinned split reductive groups over E: it has characters in $\Lambda^\sharp$, roots in $\Phi^\sharp$, simple roots in $\Delta^\sharp$, etc.

By this definition, $\hat{\mathrm{Z}}_{\mathrm{H}}$ is naturally isomorphic to $\Lambda^\sharp/\Lambda^{\sharp,r}$, so the data $(\mathrm{F}^\sharp, \varphi)$ of §1.6.5 may be interpreted as an $\mathbb{E}_\infty$-monoidal morphism $\hat{\mathrm{Z}}_{\mathrm{H}} \to \mathrm{B}^2_{\mathrm{X}}(\underline{\mathrm{A}})$. This concludes the definition of (H, F) alluded to in §1.6.3. □

1.6.8. In order to define the L-group, we need to extract the component of F which is $\mathbb{Z}$-linear, i.e. corresponding to a morphism of complexes $\hat{\mathrm{Z}}_{\mathrm{H}} \to \underline{\mathrm{A}}[2]$ of étale sheaves of abelian groups over X (see [Zha22, §6.2]).

1.6.9. Let us perform this construction in a more abstract setting: M denotes an étale sheaf of abelian groups over X. An $\mathbb{E}_\infty$-monoidal morphism $\mathrm{F} : \mathrm{M} \to \mathrm{B}^2(\underline{\mathrm{A}})$ corresponds to a symmetric monoidal extension $\widetilde{\mathrm{M}}$ of M by $\mathrm{B}(\underline{\mathrm{A}})$.

Associating to each $m \in \widetilde{\mathrm{M}}$ the commutativity constraint of $m \otimes m$ defines a character of M valued in the subgroup $\underline{\mathrm{A}}_{[2]} \subset \underline{\mathrm{A}}$ of 2-torsion elements. This character vanishes $\Leftrightarrow$ $\widetilde{\mathrm{M}}$ is strictly commutative $\Leftrightarrow$ F is $\mathbb{Z}$-linear.

Therefore, we have a fiber sequence:

$$\mathrm{Maps}_{\mathbb{Z}}(\mathrm{M}, \mathrm{B}^2(\underline{\mathrm{A}})) \to \mathrm{Maps}_{\mathbb{E}_\infty}(\mathrm{M}, \mathrm{B}^2(\underline{\mathrm{A}})) \to \mathrm{Hom}(\mathrm{M}, \underline{\mathrm{A}}_{[2]}) \tag{1.23}$$

1.6.10. The fiber sequence (1.23) canonically splits.

Construction. If $\mathrm{A}_{[2]} \neq 0$, there is nothing to construct. If $\mathrm{A}_{[2]} \neq 0$, the inclusion $\mathrm{A} \subset \mathrm{E}^\times$ identifies $\mathrm{A}_{[2]}$ with $\mathbb{Z}/2$.

Given a homomorphism $\epsilon : \mathrm{M} \to \underline{\mathrm{A}}_{[2]}$, we define the $\mathbb{E}_\infty$-monoidal morphism ${}^\epsilon\mathrm{F} : \mathrm{M} \to \mathrm{B}^2(\underline{\mathrm{A}})$ to be the trivial $\mathbb{E}_1$-monoidal morphism with commutativity constraint:

$$(\widetilde{\mathrm{M}} \ni m_1, m_2) \mapsto (-1)^{\epsilon(m_1)\epsilon(m_2)}$$

on the associated monoidal extension $\widetilde{\mathrm{M}}$ of M by $\mathrm{B}(\underline{\mathrm{A}})$. □

1.6.11. Along the split fiber sequence (1.23), F decomposes as the product:

$$\mathrm{F} \cong {}^0\mathrm{F} \otimes {}^{\mathrm{Q}}\mathrm{F}, \tag{1.24}$$

where ${}^0\mathrm{F} : \hat{\mathrm{Z}}_{\mathrm{H}} \to \mathrm{B}^2_{\mathrm{X}}(\underline{\mathrm{A}})$ is $\mathbb{Z}$-linear and ${}^{\mathrm{Q}}\mathrm{F} : \hat{\mathrm{Z}}_{\mathrm{H}} \to \mathrm{B}^2_{\mathrm{X}}(\underline{\mathrm{A}})$ is associated to the character $\mathrm{Q} \in \mathrm{Hom}(\hat{\mathrm{Z}}_{\mathrm{H}}, \underline{\mathrm{A}}_{[2]})$. (The fact that b vanishes over $\Lambda^\sharp$ implies that Q is 2-torsion valued.)

Being $\mathbb{Z}$-linear, ${}^0\mathrm{F}$ is equivalent to a global section of the complex $(\hat{\mathrm{Z}}_{\mathrm{H}})^* \otimes \underline{\mathrm{A}}[2]$ over X. Here, $(\hat{\mathrm{Z}}_{\mathrm{H}})^*$ denotes the $\mathbb{Z}$-linear dual of $\hat{\mathrm{Z}}_{\mathrm{H}}$ as a complex.

Inducing along the inclusion $\mathrm{A} \subset \mathrm{E}^\times$ and replacing E by a finite extension if necessary, ${}^0\mathrm{F}$ determines a global section of:

$$(\hat{\mathrm{Z}}_{\mathrm{H}})^* \otimes \underline{\mathrm{E}}^\times[2] \cong \mathrm{Z}_{\mathrm{H}}(\mathrm{E})[2],$$

i.e. an étale $\mathrm{Z}_{\mathrm{H}}(\mathrm{E})$-gerbe over X.

Remark 1.6.12. In what follows, we will use the same notation F (resp. ${}^0\mathrm{F}$) for the $\mathbb{E}_\infty$-monoidal morphism $\hat{\mathrm{Z}}_{\mathrm{H}} \to \mathrm{B}^2_{\mathrm{X}}(\underline{\mathrm{E}}^\times)$ (resp. $\mathbb{Z}$-linear morphism $\hat{\mathrm{Z}}_{\mathrm{H}} \to \mathrm{B}^2_{\mathrm{X}}(\underline{\mathrm{E}}^\times)$, or section of $\mathrm{Z}_{\mathrm{H}}(\mathrm{E})[2]$) induced along $\mathrm{A} \subset \mathrm{E}^\times$.

1.6.13. Let us now assume that X is connected, Noetherian, geometrically unibranch, and satisfies condition (1.1). Fix a geometric point $\bar{x}$ of X.

By taking the fiber at $\bar{x}$, we obtain a pinned split reductive group $\mathrm{H}_{\bar{x}}$ over E equipped with a $\pi_1(\mathrm{X}, \bar{x})$-action preserving the pinning.

Since $\hat{\mathrm{Z}}_{\mathrm{H}}$ is finitely generated, ${}^0\mathrm{F}$ is trivial over a finite étale cover of X. Fixing a rigidification $\bar{f}$ of ${}^0\mathrm{F}$ along $\bar{x}$ and applying the construction of §1.1.7 to finite subgroups of $\mathrm{E}^\times$, we find an extension of topological groups:

$$1 \to \mathrm{Z}_{\mathrm{H},\bar{x}}(\mathrm{E}) \to \pi_1({}^0\mathrm{F}, \bar{f}) \to \pi_1(\mathrm{X}, \bar{x}) \to 1, \tag{1.25}$$

where the $\pi_1({}^0\mathrm{F}, \bar{f})$-action on $\mathrm{Z}_{\mathrm{H},\bar{x}}(\mathrm{E})$ factors through the given $\pi_1(\mathrm{X}, \bar{x})$-action.

Inducing (1.25) along the $\pi_1(\mathrm{X}, \bar{x})$-equivariant inclusion $\mathrm{Z}_{\mathrm{H},\bar{x}}(\mathrm{E}) \subset \mathrm{H}_{\bar{x}}(\mathrm{E})$, we obtain an extension of topological groups, to be referred to as the *L-group* of (G, μ) over X:

$$1 \to \mathrm{H}_{\bar{x}}(\mathrm{E}) \to {}^{\mathrm{L}}\mathrm{H}_{\mathrm{X}} \to \pi_1(\mathrm{X}, \bar{x}) \to 1. \tag{1.26}$$

Remark 1.6.14. By construction, ${}^{\mathrm{L}}\mathrm{H}_{\mathrm{X}}$ is induced from a finite quotient $\pi_1(\mathrm{X}, \bar{x}) \twoheadrightarrow \Gamma$ and the corresponding extension ${}^{\mathrm{L}}\mathrm{H}_{\Gamma}$ of Γ by $\mathrm{H}(\mathrm{E})$ can be equipped with an algebraic structure with neutral component H.

The main difference between ${}^{\mathrm{L}}\mathrm{H}_{\mathrm{X}}$ and the L-group of a reductive group is that the quotient map ${}^{\mathrm{L}}\mathrm{H}_{\mathrm{X}} \to \pi_1(\mathrm{X}, \bar{x})$ is not equipped with a canonical section.

1.6.15. Suppose that S is the spectrum of a field k with $\mathrm{char}(k) \neq 2$ and X is a smooth, geometrically connected curve over k.

Let ω_{X} be the sheaf of differentials of X relative to k. We write $\omega_{\mathrm{X}}^{1/2}$ for the $\{\pm 1\}$-gerbe of square roots of ω_{X}. Inducing along the inclusion $\{\pm 1\} \subset \mathrm{A}$, we obtain an A-gerbe whose square is canonically trivialized, i.e. a $\mathbb{Z}$-linear morphism $\frac{1}{2}\mathbb{Z}/\mathbb{Z} \to \mathrm{B}^2_{\mathrm{X}}(\underline{\mathrm{A}})$.

Viewing Q as a $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$-valued character on $\hat{\mathrm{Z}}_{\mathrm{H}}$, we obtain a $\mathbb{Z}$-linear morphism:

$$\omega_{\mathrm{X}}^{\mathrm{Q}} : \hat{\mathrm{Z}}_{\mathrm{H}} \to \mathrm{B}^2_{\mathrm{X}}(\underline{\mathrm{A}}). \tag{1.27}$$

1.6.16. Under the assumptions of §1.6.15, we use the notation F_{ϑ} (resp. ${}^0\mathrm{F}_{\vartheta}$) for the product of F (resp. ${}^0\mathrm{F}$) with $\omega_{\mathrm{X}}^{\mathrm{Q}}$. The subscript is interpreted as a "twist by the ϑ-characteristic."

After fixing a geometric point $\bar{x}$ of X and a rigidification of (1.27) along $\bar{x}$, the $\mathbb{Z}$-linear morphism ${}^0\mathrm{F}_{\vartheta}$ defines a extension:

$$1 \to \mathrm{H}_{\bar{x}}(\mathrm{E}) \to {}^{\mathrm{L}}\mathrm{H}_{\mathrm{X},\vartheta} \to \pi_1(\mathrm{X}, \bar{x}) \to 1. \tag{1.28}$$

It is also induced along $\mathrm{Z}_{\mathrm{H},\bar{x}}(\mathrm{E}) \subset \mathrm{H}_{\bar{x}}(\mathrm{E})$ from the Baer sum of (1.25) and the central extension defined by (1.27).

These constructions have obvious analogues when X is replaced by the spectra of its field of fractions F, its local fields F_x and rings of integers $\mathscr{O}_x$.

If k satisfies $\mathrm{char}(k) = 2$, or if the 2-torsion subgroup of A is trivial, we define $\mathrm{F}_{\vartheta} = \mathrm{F}$, ${}^0\mathrm{F}_{\vartheta} = \mathrm{F}$, and ${}^{\mathrm{L}}\mathrm{H}_{\mathrm{X},\vartheta} = {}^{\mathrm{L}}\mathrm{H}_{\mathrm{X}}$.

Remark 1.6.17. In §4.2 below, we shall explain the relationship between $\omega_{\mathrm{X}}^{1/2}$ and Weissman's meta-Galois group as defined in [Wei18, §4].

In particular, when X is the field of fractions of a curve over a finite field, it will follow that (1.28) coincides with Weissman's L-group when μ comes from algebraic K-theory.

1.7. Twisted H-representations.

1.7.1. We work over a base scheme S, a finite abelian group A, and a coefficient field E as in §1.2.1. Let X be an S-scheme.

Let $\mathrm{H} \to \mathrm{X}$ be a locally constant étale sheaf of pinned split reductive groups over E. Write $\mathrm{Z}_{\mathrm{H}} \subset \mathrm{H}$ for its center and $\hat{\mathrm{Z}}_{\mathrm{H}}$ its character group, viewed as an étale sheaf of abelian groups over X. Suppose that we are supplied an $\mathbb{E}_{\infty}$-monoidal morphism $\mathrm{F} : \hat{\mathrm{Z}}_{\mathrm{H}} \to \mathrm{B}^2_{\mathrm{X}}(\underline{\mathrm{A}})$.

In this context, we shall define an étale stack of tensor categories $\mathrm{Rep}^{\{1\}}_{\mathrm{H},\mathrm{F}}$ on X. By "tensor category", we mean an E-linear symmetric monoidal abelian category.

1.7.2. Denote by $\mathrm{Lis}_{\mathrm{X}}$ the tensor category of lisse E-sheaves on X.

Note that $\mathscr{O}_{\mathrm{H}}$ may be viewed as a Hopf algebra in $\mathrm{Ind}(\mathrm{Lis}_{\mathrm{X}})$. In particular, there is the notion of an H-*representation on a lisse* E-*sheaf*: it is an object $\mathscr{F} \in \mathrm{Lis}_{\mathrm{X}}$ equipped with a morphism $\mathscr{F} \to \mathscr{F} \otimes \mathscr{O}_{\mathrm{H}}$ in $\mathrm{Ind}(\mathrm{Lis}_{\mathrm{X}})$ satisfying the axioms defining a coaction.

Let $\mathrm{Rep}_{\mathrm{H}}^{\{1\}}$ denote the tensor category of H-representations on lisse E-sheaves over X. The forgetful functor $\mathrm{Rep}_{\mathrm{H}}^{\{1\}} \to \mathrm{Lis}_{\mathrm{X}}$ is E-linear and symmetric monoidal.

1.7.3. The construction $\mathrm{X} \mapsto \mathrm{Rep}_{\mathrm{H}}^{\{1\}}$ being of étale local nature, we obtain a stack of tensor categories $\underline{\mathrm{Rep}}_{\mathrm{H}}^{\{1\}}$ on the étale site of X. It admits a decomposition according to the weights of the Z_{H}-action, compatible with the tensor structure:

$$\underline{\mathrm{Rep}}_{\mathrm{H}}^{\{1\}} \cong \bigoplus_{\lambda \in \hat{\mathrm{Z}}_{\mathrm{H}}} \underline{\mathrm{Rep}}_{\mathrm{H}}^{\{1\},\lambda}. \tag{1.29}$$

Since A acts by automorphisms of the identity endofunctor of $\underline{\mathrm{Rep}}_{\mathrm{H}}^{\{1\}}$, we may form the F-twisted stack of tensor categories $\underline{\mathrm{Rep}}_{\mathrm{H,F}}^{\{1\}}$ as in [Zha22, Appendix A].

Finally, we define $\mathrm{Rep}_{\mathrm{H,F}}^{\{1\}}$ to be the global section of $\underline{\mathrm{Rep}}_{\mathrm{H,F}}^{\{1\}}$.

Remark 1.7.4. There is a decomposition inherited from (1.29):

$$\mathrm{Rep}_{\mathrm{H,F}}^{\{1\}} \cong \bigoplus_{\lambda \in \check{\mathrm{Z}}_{\mathrm{H}}} (\mathrm{Rep}_{\mathrm{H}}^{\{1\},\lambda})_{\mathrm{F}(\lambda)}, \tag{1.30}$$

where each summand is the $\mathrm{F}(\lambda)$-twist of the abelian E-linear categories $\mathrm{Rep}_{\mathrm{H}}^{\{1\},\lambda}$.

The symmetric monoidal structure on $\mathrm{Rep}_{\mathrm{H,F}}^{\{1\}}$ is induced from that of $\mathrm{Rep}_{\mathrm{H}}^{\{1\}}$ and the $\mathbb{E}_\infty$-monoidal structure of F. Concretely, the monoidal product is given by:

$$\begin{aligned}(\mathrm{Rep}_{\mathrm{H}}^{\{1\},\lambda_1})_{\mathrm{F}(\lambda_1)} \times (\mathrm{Rep}_{\mathrm{H}}^{\{1\},\lambda_2})_{\mathrm{F}(\lambda_2)} &\to (\mathrm{Rep}_{\mathrm{H}}^{\{1\},\lambda_1+\lambda_2})_{\mathrm{F}(\lambda_1)\otimes\mathrm{F}(\lambda_2)} \\ &\cong (\mathrm{Rep}_{\mathrm{H}}^{\{1\},\lambda_1+\lambda_2})_{\mathrm{F}(\lambda_1+\lambda_2)}.\end{aligned}$$

Remark 1.7.5. Suppose that X is connected, Noetherian, geometrically unibranch, and satisfies condition (1.1). Suppose also that F is $\mathbb{Z}$-linear. (In practice, F will be one of the objects ${}^0\mathrm{F}$, ${}^0\mathrm{F}_\vartheta$ defined in §1.6.)

We fix a geometric point $\bar{x}$ of X and a rigidification $\bar{f}$ of F along $\bar{x}$. Then the L-group (1.26) can be used to give a "hands-on" description of $\mathrm{Rep}_{\mathrm{H,F}}^{\{1\}}$.

Indeed, given a profinite group Γ and an extension ${}^{\mathrm{L}}\mathrm{H}$ of Γ by $\mathrm{H}_{\bar{x}}(\mathrm{E})$, we write $\mathrm{Rep}_{{}^{\mathrm{L}}\mathrm{H}}^{\mathrm{alg}}$ for the category of finite-dimensional continuous representations of ${}^{\mathrm{L}}\mathrm{H}$ whose restriction along $\mathrm{H}_{\bar{x}}(\mathrm{E}) \subset {}^{\mathrm{L}}\mathrm{H}$ lifts to an algebraic representation of $\mathrm{H}_{\bar{x}}$.

Applied to ${}^{\mathrm{L}}\mathrm{H}_{\mathrm{X}}$, this contruction yields a tensor category canonically equivalent to that of F-twisted H-representations:

$$\mathrm{Rep}_{{}^{\mathrm{L}}\mathrm{H}_{\mathrm{X}}}^{\mathrm{alg}} \cong \mathrm{Rep}_{\mathrm{H,F}}^{\{1\}}. \tag{1.31}$$

Under the equivalence (1.31), the underlying (algebraic) $\mathrm{H}_{\bar{x}}$-representation of an object $\mathrm{V} \in \mathrm{Rep}_{{}^{\mathrm{L}}\mathrm{H}_{\mathrm{X}}}^{\mathrm{alg}}$ is isomorphic to the fiber of the corresponding object in $\mathrm{Rep}_{\mathrm{H,F}}^{\{1\}}$ at $\bar{x}$, by passing through the rigidification $\bar{f}$.

1.7.6. The construction of $\mathrm{Rep}_{\mathrm{H,F}}^{\{1\}}$ has a multiple-point generalization, which justifies the superscript in the notation.

Indeed, for a nonempty finite set I, we have an $\mathbb{E}_\infty$-monoidal morphism of étale sheaves over X^{I} out of an external direct sum of copies of $\hat{\mathrm{Z}}_{\mathrm{H}}$:

$$\mathrm{F}^{\mathrm{I}} : \hat{\mathrm{Z}}_{\mathrm{H}}^{\mathrm{I}} \to \mathrm{B}^2_{\mathrm{X}^{\mathrm{I}}}(\underline{\mathrm{A}}), \quad (\lambda_i)_{i\in\mathrm{I}} \mapsto \boxtimes_{i\in\mathrm{I}} \mathrm{F}(\lambda_i). \tag{1.32}$$

Viewing H^{I} as a locally constant étale sheaf over X^{I} of pinned split reductive groups over E, the corresponding étale sheaf $\hat{\mathrm{Z}}_{\mathrm{H}^{\mathrm{I}}}$ is identified with $\hat{\mathrm{Z}}_{\mathrm{H}}^{\mathrm{I}}$. The construction of §1.7.3, applied to X^I, H^I, and F^I, yields a stack of tensor categories $\underline{\mathrm{Rep}}^I_{H^I,F^I}$ over X^I, and we set $\mathrm{Rep}^I_{H^I,F^I}$ to be its global section.

1.7.7. The $\mathbb{E}_\infty$-monoidal structure on F^I (for varying I) induces its compatibility data with respect to restrictions along the diagonals.

More precisely, given a surjection of nonempty finite sets $p : I \twoheadrightarrow J$, giving rise to the diagonal immersion $\Delta^p : X^J \to X^I$, we obtain an isomorphism $(\Delta^p)^* F^I \cong F^J$. These isomorphisms are compatible with compositions in the obvious sense.

Correspondingly the association $(I \neq \varnothing) \mapsto \mathrm{Rep}^I_{H^I,F^I}$ defines a functor from the category of nonempty finite sets with surjections to the 2-category of tensor categories. It carries a surjection $p : I \twoheadrightarrow J$ to the composition:

$$\mathrm{Rep}^I_{H^I,F^I} \to \mathrm{Rep}^J_{(\Delta^p)^*(H^I),F_J} \to \mathrm{Rep}^J_{H^J,F_J} \tag{1.33}$$

where the first functor is the restriction along Δ^p, whereas the second functor is the restriction of the action along the diagonal $H^J \to (\Delta^p)^*(H^I)$.

2. Geometric Satake equivalence

The goal of this section is to state the geometric Satake equivalence for étale metaplectic covers: Theorem 2.4.4. It is the metaplectic analogue of the equivalence of Mirković–Vilonen [MV07] and Gaitsgory [Gai07, Theorem 2.6].

Sections §2.1–2.3 are preparatory. The main equivalence is stated in §2.4. In §2.5, we shall use it to define a collection of functors, called "Satake functors", which play an instrumental role in the proof of the spectral decomposition theorem in §4.

2.0.1. Throughout this section, we work over a field k. The letter S is reserved for arbitrary ("test") affine k-schemes.

Let ℓ be a prime invertible in k and fix an algebraic closure $\mathbb{Q}_\ell \subset \overline{\mathbb{Q}}_\ell$. The coefficient field E will be an intermediate field $\mathbb{Q}_\ell \subset E \subset \overline{\mathbb{Q}}_\ell$.[3] Let $A \subset E^\times$ be a finite subgroup whose order is invertible in k.

Let X be a smooth curve over k and $G \to X$ be a smooth affine group scheme. Let μ be an A-valued étale metaplectic cover of G, i.e. a rigidified section of $B^4\underline{A}(1)$ over $B_X(G)$.

2.1. The local Hecke stack.

2.1.1. Let I be a nonempty finite set.

For an S-point x^I of X^I, we write Γ_{x^I} for the scheme-theoretic union of the graphs of $x^i : S \to X$ (over $i \in I$), D_{x^I} for the formal completion of $S \times X$ along Γ_{x^I}, and $\mathring{D}_{x^I}$ the open subscheme $D_{x^I} \backslash \Gamma_{x^I}$. We call D_{x^I} (resp. $\mathring{D}_{x^I}$) the *formal disk* (resp. *formal punctured disk*) around Γ_{x^I}.

We shall define a number of étale stacks over X^I. Their groupoids of lifts of an S-point x^I of X^I are tabulated below:

$L^I_+(G)$	a section of G over D_{x^I}	
$L^I(G)$	a section of G over $\mathring{D}_{x^I}$	
Gr^I_G	a G-torsor P over D_{x^I} equipped with $\alpha : P^0 \overset{x^I}{\sim} P$	(2.1)
Hec^I_G	G-torsors P_0, P_1 over D_{x^I} equipped with $\alpha : P_0 \overset{x^I}{\sim} P_1$	

[3] It would be interesting to treat integral cofficients, but I have not attempted to do so.

Here, P^0 stands for the trivial G-torsor over D_{x^I}, and the notation $P_0 \overset{x^I}{\sim} P_1$ for two G-torsors over D_{x^I} means an isomorphism of them off Γ_{x^I}.

The étale sheaves $L^I_+(G)$ and $L^I(G)$ are valued in groups. The étale sheaf Gr^I_G (resp. stack Hec^I_G) is ind-schematic (resp. ind-algebraic) of ind-finite type.

If $G \to X$ is reductive, Gr^I_G is furthemore ind-proper.

Remark 2.1.2. Since G is smooth, there are canonical isomorphisms $\mathrm{Gr}^I_G \cong L^I G/L^I_+ G$ and $\mathrm{Hec}^I_G \cong L^I_+ G \backslash L^I G/L^I_+ G$, where the quotients are taken in the étale topology. The quotient map $\pi : \mathrm{Gr}^I_G \to \mathrm{Hec}^I_G$ sends (P, α) to the triple (P^0, P, α).

2.1.3. Given a surjection of nonempty finite sets $p : I \twoheadrightarrow J$, we write $I_j := f^{-1}(j)$ and view p as an unordered partition of I. Denote by $X^p \subset X^I$ the "disjoint locus", i.e. the open subscheme where $\Gamma_{x^{I_{j_1}}} \cap \Gamma_{x^{I_{j_2}}} = \varnothing$ if $j_1 \neq j_2 \in J$.

We have a canonical isomorphism of étale stacks over X^p via restriction to each $D_{X^{I_j}}$:

$$\varphi^p : \mathrm{Hec}^I_G \times_{X^I} X^p \cong (\prod_{j\in J} \mathrm{Hec}^{I_j}_G) \times_{X^I} X^p. \tag{2.2}$$

The isomorphism (2.2) is compatible with refinements of partitions. Namely, given two composable surjections of nonempty finite sets $p : I \twoheadrightarrow J$, $q : J \twoheadrightarrow K$, we have containments $X^p \subset X^{q\cdot p} \subset X^I$. For each $k \in K$, p restricts to the partition $p_k : I_k := (q \cdot p)^{-1}(k) \twoheadrightarrow J_k$. There is a commutative diagram:

$$\begin{array}{ccc} \mathrm{Hec}^I_G \times_{X^I} X^p & \xrightarrow{\varphi^{q\cdot p}} & (\prod_{k\in K} \mathrm{Hec}^{I_k}_G) \times_{X^I} X^p \\ \downarrow{\scriptstyle \varphi^p} & & \downarrow{\scriptstyle \prod_{k\in K} \varphi^{p_k}} \\ (\prod_{j\in J} \mathrm{Hec}^{I_j}_G) \times_{X^I} X^p & \cong & \prod_{k\in K}(\prod_{j\in J_k} \mathrm{Hec}^{I_j}_G) \times_{X^I} X^p \end{array} \tag{2.3}$$

Furthermore, given three composable surjections of nonempty finite sets, the witness of commutativity of (2.3) satisifes the evident cocycle condition.

The isomorphism (2.2) and the commutativity (2.3) satisfying the cocycle condition are referred to as the *factorization structure* of Hec^I_G (for varying I).

Remark 2.1.4. If J is equipped with an ordering, or equivalently if I admits an ordered partition $I = I_1 \sqcup \cdots \sqcup I_k$, then we change the notation X^p to $X^{I_1,\cdots,I_k}$ and analogously for the base change of the prestacks in (2.1): $\mathrm{Hec}^{I_1,\cdots,I_k}_G := \mathrm{Hec}^I_G \times_{X^I} X^{I_1,\cdots,I_k}$

2.1.5. Given an *ordered* partition $I = I_1 \sqcup \cdots \sqcup I_k$ as nonempty finite sets, we shall consider another étale stack $\widetilde{\mathrm{Hec}}^{I_1,\cdots,I_k}_G$ over X^I whose lift of an S-point x^I of X^I consists of G-torsors $P_0, \cdots, P_k$ together with isomorphisms:

$$(P_0 \overset{x^{I_1}}{\sim} P_1 \overset{x^{I_2}}{\sim} \cdots \overset{x^{I_k}}{\sim} P_k). \tag{2.4}$$

The functors p_a of remembering each segment $(x^{I_a}, P_{a-1} \overset{x^{I_a}}{\sim} P_a)$ (over $1 \le a \le k$) and the functor m of remebering their composition $(x^I, P_0 \overset{x^I}{\sim} P_k)$ define two morphisms:

$$\begin{array}{ccc} \widetilde{\mathrm{Hec}}^{I_1,\cdots,I_k}_G & \xrightarrow{\prod p_a} & \prod_{1\le a\le k} \mathrm{Hec}^{I_a}_G \\ \downarrow{\scriptstyle m} & & \\ \mathrm{Hec}^I_G & & \end{array} \tag{2.5}$$

Over the disjoint locus $X^{I_1,\cdots,I_k}$, both maps in (2.5) restrict to isomorphisms and their composition is identified with the factorization isomorphism (2.2) associated to the underlying unordered partition $I \to \{1,\cdots,k\}$.

This construction has an obvious analogue for the affine Grassmannian, namely an ind-scheme $\widetilde{\mathrm{Gr}}_G^{I_1,\cdots,I_k} \to X^I$ classifying the same data (2.4) as $\widetilde{\mathrm{Hec}}_G^{I_1,\cdots,I_k}$, together with an additional trivialization of P_0.

2.1.6. Given a nonempty finite set I, the association:

$$([k] \in \Delta^{\mathrm{op}}) \mapsto \widetilde{\mathrm{Hec}}_G^{I,[k]} := \widetilde{\mathrm{Hec}}_G^{I_1,\cdots,I_k} \times_{X^{I_1 \sqcup \cdots \sqcup I_k}} X^I, \tag{2.6}$$

with $I_1 = \cdots = I_k := I$ defines a simplicial étale stack over X^I.

By convention, we set $\widetilde{\mathrm{Hec}}_G^{I,[0]} := B_{X^I}(L_+^I(G))$, i.e. the stack classifying a G-torsor P over D_{x^I} with no modifications. Morphisms in Δ^{op} are carried to compositions of the corresponding segment of modifications in (2.4).

This simplicial étale stack is canonically identified with the Čech nerve of the morphism:

$$B_{X^I}(L_+^I(G)) \to B_{X^I}(L^I(G)). \tag{2.7}$$

Evidently, its value at [1] is Hec_G^I. In particular, the simplicial system (2.6) may be viewed as an additional structure on Hec_G^I which we call the *convolution structure*.

2.2. The local A-gerbe.

2.2.1. Let I be a nonempty finite set.

We shall use the étale metaplectic cover μ to define an A-gerbe $\mathscr{G}^I$ on Hec_G^I. Furthermore, $\mathscr{G}^I$ (for varying I) will come equipped with canonical compatibility data with respect to the factorization and convolution structures of Hec_G^I.

The A-gerbe $\mathscr{G}^I$ is not new: it features prominently in [Rei12] and [GL18], and we will explain in Remark 2.2.8 how their approach is related to ours.

2.2.2. We start by defining an $\mathbb{E}_1$-monoidal section of $B^2(\underline{A})$ over $L^I(G)$, trivialized as such over $L_+^I(G)$, *i.e.* a commutative diagram of $\mathbb{E}_1$-monoidal morphisms:

$$\begin{array}{ccc} L_+^I(G) & \longrightarrow & X^I \\ \downarrow & & \downarrow{\scriptstyle e} \\ L^I(G) & \longrightarrow & B_{X^I}^2(\underline{A}) \end{array} \tag{2.8}$$

Construction. For an S-point x^I of X^I, we give names to the natural morphisms in the following diagram as displayed:

$$\begin{array}{ccccc} \Gamma_{x^I} & \xrightarrow{\hat{i}} & D_{x^I} & \xleftarrow{\hat{j}} & \mathring{D}_{x^I} \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow \\ \Gamma_{x^I} & \xrightarrow{i} & S \times X & \xleftarrow{j} & S \times X \backslash \Gamma_{x^I} \\ & & \downarrow{\scriptstyle p} & & \\ & & S & & \end{array}$$

Viewing μ as an $\mathbb{E}_1$-monoidal morphism $G \to B_X^3\underline{A}(1)$, we obtain an $\mathbb{E}_1$-monoidal morphism by taking its section over $\mathring{D}_{x^I} \to X$:

$$\mu_* : \Gamma(\mathring{D}_{x^I}, G) \to \Gamma(\mathring{D}_{x^I}, B_X^3\underline{A}(1)). \tag{2.9}$$

Note that the target is the ∞-groupoid associated to the connective truncation of the complex $\Gamma(\mathrm{D}_{x^{\mathrm{I}}}, \hat{j}_* \underline{\mathrm{A}}(1)[3])$.

Using the Cousin complex and the Gabber–Fujiwara formal base change theorem [Fuj95, Corollary 6.6.4] (see [BM21, Theorem 6.11] for a proof avoiding the Noetherian hypothesis), we find morphisms of complexes:

$$\Gamma(\mathrm{D}_{x^{\mathrm{I}}}, \hat{j}_* \underline{\mathrm{A}}(1)[3]) \to \Gamma(\Gamma_{x^{\mathrm{I}}}, \hat{i}^! \underline{\mathrm{A}}(1)[4]) \\ \cong \Gamma(\Gamma_{x^{\mathrm{I}}}, i^! \underline{\mathrm{A}}(1)[4]) \cong \Gamma(\Gamma_{x^{\mathrm{I}}}, (p \cdot i)^! \underline{\mathrm{A}}[2]) \to \Gamma(\mathrm{S}, \underline{\mathrm{A}}[2]), \tag{2.10}$$

where the last two maps use the smoothness of p and the properness of $p \cdot i$, respectively.

Composing (2.9) with the morphism on underlying ∞-groupoids of (2.10), we obtain an $\mathbb{E}_1$-monoidal morphism:

$$\Gamma(\mathring{\mathrm{D}}_{x^{\mathrm{I}}}, \mathrm{G}) \to \Gamma(\mathrm{S}, \mathrm{B}^2(\underline{\mathrm{A}})), \tag{2.11}$$

trivialized as such over $\Gamma(\mathrm{D}_{x^{\mathrm{I}}}, \mathrm{G})$. The construction being functorial in the S-point x^{I}, we obtain the commutative diagram (2.8). ☐

Remark 2.2.3. Fix a closed point $x \in \mathrm{X}$ and write $\mathrm{L}(\mathrm{G})_x$ for the fiber of $\mathrm{L}^{\{1\}}(\mathrm{G})$ at x.

The construction of the $\mathbb{E}_1$-monoidal morphism $\mathrm{L}(\mathrm{G})_x \to \mathrm{B}^2_x(\underline{\mathrm{A}})$ in §2.2.2 only requires μ to be defined over $\mathring{\mathrm{D}}_x$ (instead of X). Furthermore, it is $\mathbb{E}_1$-monoidally trivial over the first congruence group scheme:

$$\mathrm{G}_{1,x} := \mathrm{Ker}(\mathrm{L}_+(\mathrm{G})_x \to \mathrm{G}_x),$$

because $\mathrm{G}_{1,x}$ is pro-unipotent. (However, lifting this trivialization to one over $\mathrm{L}_+(\mathrm{G})_x$ in general requires μ to be defined over D_x.)

2.2.4. The commutative diagram (2.8) is equivalent to a section of $\mathrm{B}^3(\underline{\mathrm{A}})$ over $\mathrm{B}_{\mathrm{X}^{\mathrm{I}}}(\mathrm{L}^{\mathrm{I}}(\mathrm{G}))$ trivialized over $\mathrm{B}_{\mathrm{X}^{\mathrm{I}}}(\mathrm{L}^{\mathrm{I}}_+(\mathrm{G}))$.

Taking Čech nerves, we obtain a morphism of simplicial étale stacks:

$$\widetilde{\mathrm{Hec}}^{\mathrm{I},[k]}_{\mathrm{G}} \to \mathrm{B}^2(\mathrm{A})^{\times k}, \quad [k] \in \Delta^{\mathrm{op}}, \tag{2.12}$$

where the target $[k] \mapsto \mathrm{B}^2(\mathrm{A})^{\times k}$ is the Čech nerve of the morphism $\mathrm{Spec}(k) \to \mathrm{B}^3(\mathrm{A})$.

Finally, we define $\mathscr{G}^{\mathrm{I}}$ to be the value of (2.12) at [1].

Remark 2.2.5. We view the simplicial morphism (2.12) as expressing the compatibility between $\mathscr{G}^{\mathrm{I}}$ and the convolution structure of $\mathrm{Hec}^{\mathrm{I}}_{\mathrm{G}}$.

For instance, the commutation of (2.12) with the three face maps $([1] \to [2]) \in \Delta$ contain the following isomorphism of A-gerbes on $\widetilde{\mathrm{Hec}}^{\mathrm{I},[2]}_{\mathrm{G}}$:

$$m^*(\mathscr{G}^{\mathrm{I}}) \cong p_1^*(\mathscr{G}^{\mathrm{I}}) \otimes p_2^*(\mathscr{G}^{\mathrm{I}}), \tag{2.13}$$

where the morphisms p_2, m, p_1 send an S-point $(\mathrm{P}_0 \overset{x^{\mathrm{I}}}{\sim} \mathrm{P}_1 \overset{x^{\mathrm{I}}}{\sim} \mathrm{P}_2)$ of $\widetilde{\mathrm{Hec}}^{\mathrm{I},[2]}_{\mathrm{G}}$ to S-points $(\mathrm{P}_1 \overset{x^{\mathrm{I}}}{\sim} \mathrm{P}_2)$, $(\mathrm{P}_0 \overset{x^{\mathrm{I}}}{\sim} \mathrm{P}_2)$, respectively $(\mathrm{P}_0 \overset{x^{\mathrm{I}}}{\sim} \mathrm{P}_1)$, of $\mathrm{Hec}^{\mathrm{I}}_{\mathrm{G}}$.

The commutation of (2.12) with the degeneracy map $([1] \to [0]) \in \Delta$ expresses the fact that $\mathscr{G}^{\mathrm{I}}$ is canonically rigidified along the unit $e : \mathrm{B}_{\mathrm{X}^{\mathrm{I}}}(\mathrm{L}^{\mathrm{I}}_+(\mathrm{G})) \to \mathrm{Hec}^{\mathrm{I}}_{\mathrm{G}}$.

2.2.6. We note a variant of the compatibility between $\mathscr{G}^{\mathrm{I}}$ and the convolution structure, for the Hecke stack $\widetilde{\mathrm{Hec}}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}_{\mathrm{G}}$ associated to an ordered partition $\mathrm{I} = \mathrm{I}_1 \sqcup \cdots \sqcup \mathrm{I}_k$.

Namely, along the two morphisms of (2.5), we have a canonical isomorphism of A-gerbes on $\widetilde{\mathrm{Hec}}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}_{\mathrm{G}}$:

$$m^*(\mathscr{G}^{\mathrm{I}}) \cong \bigotimes_{1 \le a \le k} p_a^*(\mathscr{G}^{\mathrm{I}_a}). \tag{2.14}$$

To see this, we express $\widetilde{\mathrm{Hec}}_{\mathrm{G}}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}$ as the quotient of the group stack $\widetilde{\mathrm{L}}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}(\mathrm{G}) \to \mathrm{X}^{\mathrm{I}}$, whose lift of an S-point x^{I} of X^{I} consists of sections g_a of G over $\mathrm{D}_{x^{\mathrm{I}}}\backslash\Gamma_{x^{\mathrm{I}_a}}$ (for $1 \le a \le k$), by the actions of $(k+1)$ copies of $\mathrm{L}^{\mathrm{I}}_+\mathrm{G}$ via the fomulas:

$$h\cdot(g_1,\cdots,g_k) = \begin{cases}(hg_1,g_2,\cdots,g_k)\\ (g_1h^{-1},hg_2,\cdots,g_k)\\ \quad\vdots \\ (g_1,\cdots,g_{k-1},g_kh^{-1})\end{cases}$$

Given an S-point $(x^{\mathrm{I}},g_1,\cdots,g_k)$ of $\widetilde{\mathrm{L}}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}(\mathrm{G})$, the section in $\Gamma(\Gamma_{x^{\mathrm{I}}}, i^!\underline{\mathrm{A}}(1))$ defined by the sum of the sections in $\Gamma(\Gamma_{x^{\mathrm{I}_a}}, i^!_{x^{\mathrm{I}_a}}\underline{\mathrm{A}}(1))$ (for $i_{x^{\mathrm{I}_a}} : \Gamma_{x^{\mathrm{I}_a}} \to \mathrm{S}\times\mathrm{X}$ the closed immersion) via (2.9) is identified with the section defined by $(x^{\mathrm{I}}, g_1\cdots g_k) \in \mathrm{L}^{\mathrm{I}}(\mathrm{G})$. This identification induces the isomorphism (2.14).

The isomorphism (2.14) is compatible with refinements of the ordered partition of I, in the evident sense.

2.2.7. Finally, the A-gerbe $\mathscr{G}^{\mathrm{I}}$ is also compatible with the factorization structure on $\mathrm{Hec}^{\mathrm{I}}_{\mathrm{G}}$ (for varying I) in the following sense: (2.2) is upgraded to an isomorphism in the 2-category of prestacks equipped with an A-gerbe (see §1.2.6) where the A-gerbe over $\mathrm{Hec}^{\mathrm{I}}_{\mathrm{G}}$ is $\mathscr{G}^{\mathrm{I}}$ and the A-gerbe over $\prod_{j\in\mathrm{J}}\mathrm{Hec}^{\mathrm{I}_j}_{\mathrm{G}}$ is the external product $\boxtimes_{j\in\mathrm{J}}\mathscr{G}^{\mathrm{I}_j}$.

The 2-isomorphism (2.3) and its cocycle condition also take place in the 2-category of such pairs.

Remark 2.2.8. The restriction of $\mathscr{G}^{\mathrm{I}}$ along $\mathrm{Gr}^{\mathrm{I}}_{\mathrm{G}} \to \mathrm{Hec}^{\mathrm{I}}_{\mathrm{G}}$ defines a "symmetric factorization A-gerbe" in the sense of [Rei12]. The symmetry datum is encoded by the 2-isomorphism of (2.3) corresponding to $\mathrm{I} \twoheadrightarrow \mathrm{J} \xrightarrow{q} \mathrm{J}$ where q is an automorphism.

Contrary to [Rei12] and [GL18], we do not take factorization A-gerbes as parameters for covering groups, although they turn out to be equivalent to étale metaplectic covers over a smooth curve [Zha20]. An advantage of étale metaplectic covers is that their compatibility with the convolution structure on $\mathrm{Hec}^{\mathrm{I}}_{\mathrm{G}}$ (and $\widetilde{\mathrm{Hec}}_{\mathrm{G}}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}$) is essentially tautological.

2.2.9. Suppose that k is a finite field, $\mathrm{I} = \{1\}$, and $x \in \mathrm{X}$ is a k-point.

Let us assume that μ is only defined over $\mathring{\mathrm{D}}_x \cong \mathrm{Spec}(\mathrm{F}_x)$. Recall that μ gives rise to a central extension $\widetilde{\mathrm{G}}_x$ of $\mathrm{G}(\mathrm{F}_x)$ by A (1.19). Let us recover this central extension from the A-gerbe $\mathscr{G}^{\{1\}}$ via the trace-of-Frobenius construction (see §1.4).

Indeed, write $\mathrm{L}(\mathrm{G})_x$ for the base change of $\mathrm{L}^{\{1\}}(\mathrm{G})$ to x and $\mathscr{G}_x$ for the restriction of $\mathscr{G}^{\{1\}}$ to $\mathrm{L}(\mathrm{G})_x$. Note that k-points of $\mathrm{L}(\mathrm{G})_x$ are canonically identified with $\mathrm{G}(\mathrm{F}_x)$. We shall construct an isomorphism of multiplicative A-torsors over the group $\mathrm{G}(\mathrm{F}_x)$:

$$\mathrm{Tr}(\mathrm{Fr}\mid\mathscr{G}_x)(k) \cong \widetilde{\mathrm{G}}_x. \tag{2.15}$$

Construction. By the constructions of §2.2.2 and [Zha22, §2.1], the two sides of (2.15) arise as the compositions of $\mu : \mathrm{G} \to \mathrm{B}^3\mathrm{A}(1)$ with the morphisms on $\mathbb{E}_1$-monoids induced from the two circuits of the following diagram:

$$\begin{array}{ccc}\Gamma(\mathrm{F}_x,\mathrm{A}(1)[3]) & \xrightarrow{\text{Cousin}} & \Gamma(x,\mathrm{A}[2]) \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\mathrm{Tr}(\mathrm{Fr}|-)} \\ \Gamma(\mathrm{F}_x,\mathrm{A}(1)[2])[1] & \xrightarrow{\text{Tate}} & \mathrm{A}[1]\end{array} \tag{2.16}$$

Here, the bottom horizontal arrow is induced from the Tate-duality map $\mathrm{H}^2(\mathrm{F}_x,\mathrm{A}(1)) \cong \mathrm{A}$.

It remains to note that (2.16) commutes thanks to our normalization of the Tate duality map (see §1.5.4). □

Remark 2.2.10. If $x \in \mathrm{X}$ is a closed point with residue field $k_1 \supset k$, a small modification is needed to recover the central extension $\widetilde{\mathrm{G}}_x$ from geometry.

Namely, we consider the Weil restriction $\mathrm{res}(\mathrm{L(G)}_x)$ of $\mathrm{L(G)}_x$ along $x \to \mathrm{Spec}(k)$. The A-gerbe $\mathscr{G}_x$ defines an A-gerbe $\mathrm{Nm}(\mathscr{G}_x)$ over $\mathrm{res}(\mathrm{L(G)}_x)$ (see Remark 1.4.8) and we obtain an isomorphism of multiplicative A-torsors over $\mathrm{G}(\mathrm{F}_x)$:

$$\mathrm{Tr}(\mathrm{Fr} \mid \mathrm{Nm}(\mathscr{G}_x))(k) \cong \widetilde{\mathrm{G}}_x,$$

by combining (1.17) and (2.15).

Remark 2.2.11. It follows from the identification (2.15) that $\widetilde{\mathrm{G}}_x$ is canonically split over the first congruence subgroup $\mathrm{G}_{1,x}(k)$ (Remark 2.2.3). If μ is defined over D_x, this splitting extends to one over $\mathrm{G}(\mathscr{O}_x)$ by the same remark and coincides with the one in §1.5.4.

2.3. The Satake category.

2.3.1. Let S be a k-scheme. Let Y be a separated S-scheme of finite presentation equipped with an A-gerbe $\mathscr{G}$.

Recall the ∞-category $\mathrm{Shv}_{\mathscr{G}}(\mathrm{Y})$ of $\mathscr{G}$-twisted sheaves on Y defined in §1.2.

Denote by $\mathrm{Shv}_{\mathscr{G}}(\mathrm{Y})_{/\mathrm{S}} \subset \mathrm{Shv}_{\mathscr{G}}(\mathrm{Y})$ the full ∞-subcategory of $\mathscr{G}$-twisted sheaves universally locally acyclic relative to $\mathrm{Y} \to \mathrm{S}$. The condition of universal local acyclicity is well-defined for $\mathscr{G}$-twisted sheaves because it is of étale local nature on the source.

The ∞-category $\mathrm{Shv}_{\mathscr{G}}(\mathrm{Y})$ admits a perverse t-structure *relative to* $\mathrm{Y} \to \mathrm{S}$, see [HS21, Theorem 1.1]. Let $\mathrm{Perv}_{\mathscr{G}}(\mathrm{Y})$ denote its heart. We refer to its objects simply as "perverse sheaves" on Y.

The full ∞-subcategory $\mathrm{Shv}_{\mathscr{G}}(\mathrm{Y})_{/\mathrm{S}}$ inherits a t-structure ([HS21, Theorem 6.7]). In particular, we have the abelian category:

$$\mathrm{Perv}_{\mathscr{G}}(\mathrm{Y})_{/\mathrm{S}} \coloneqq \mathrm{Perv}_{\mathscr{G}}(\mathrm{Y}) \cap \mathrm{Shv}_{\mathscr{G}}(\mathrm{Y})_{/\mathrm{S}}.$$

These notions generalize to the situation where Y is an ind-scheme of ind-finite presentation via left Kan extension.

2.3.2. Let us now assume that $\mathrm{G} \to \mathrm{X}$ is a split reductive group scheme.

Let I be a nonempty finite set. Recall the A-gerbe $\mathscr{G}^{\mathrm{I}}$ on $\mathrm{Hec}_{\mathrm{G}}^{\mathrm{I}}$ defined by μ in §2.2.

The *Satake category* is the full subcategory

$$\mathrm{Sat}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}^{\mathrm{I}} \subset \mathrm{Shv}_{\mathscr{G}^{\mathrm{I}}}(\mathrm{Hec}_{\mathrm{G}}^{\mathrm{I}})$$

characterized by the following property: an object belongs to $\mathrm{Sat}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}^{\mathrm{I}}$ if its pullback along $\pi : \mathrm{Gr}_{\mathrm{G}}^{\mathrm{I}} \to \mathrm{Hec}_{\mathrm{G}}^{\mathrm{I}}$ (see Remark 2.1.2) belongs to $\mathrm{Perv}_{\mathscr{G}^{\mathrm{I}}}(\mathrm{Gr}_{\mathrm{G}}^{\mathrm{I}})_{/\mathrm{X}^{\mathrm{I}}}$.

Remark 2.3.3. As in the non-twisted setting, $\mathrm{Sat}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}^{\mathrm{I}}$ is an abelian category and the pullback functor defines a fully faithful embedding:

$$\pi^* : \mathrm{Sat}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}^{\mathrm{I}} \subset \mathrm{Perv}_{\mathscr{G}^{\mathrm{I}}}(\mathrm{Gr}_{\mathrm{G}}^{\mathrm{I}})_{/\mathrm{X}^{\mathrm{I}}}.$$

In particular, we may view objects of $\mathrm{Sat}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}^{\mathrm{I}}$ as $\mathscr{G}^{\mathrm{I}}$-twisted universally locally acyclic perverse sheaves on $\mathrm{Gr}_{\mathrm{G}}^{\mathrm{I}}$ satisfying an extra "equivariance" condition.

2.3.4. Let us equip $\mathrm{Sat}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}$ with a symmetric monoidal structure, where the monoidal product is given either by the "convolution", or the "fusion" product.

The construction explained below is a straightforward adaptation of its non-metaplectic counterpart. We choose to follow the approach of [FS21, VI] instead of [MV07].

2.3.5. To begin with, since $\mathrm{Hec}^{\mathrm{I}}_{\mathrm{G}}$ admits a convolution structure (§2.1.6) and $\mathscr{G}^{\mathrm{I}}$ is compatible with it (§2.2.4), the ∞-category $\mathrm{Shv}_{\mathscr{G}^{\mathrm{I}}}(\mathrm{Hec}^{\mathrm{I}}_{\mathrm{G}})$ admits an $\mathbb{E}_1$-monoidal structure.

More concretely, the monoidal product is given by:

$$\mathscr{F}_1 \circ_{\mathrm{X}^{\mathrm{I}}} \mathscr{F}_2 := m_!(p_1^*\mathscr{F}_1 \otimes p_2^*\mathscr{F}_2), \tag{2.17}$$

passing through the isomorphism (2.13) of A-gerbes. The monoidal unit is given by $e_!(\underline{\mathrm{E}})$ using the rigidification of $\mathscr{G}^{\mathrm{I}}$ along e. It clearly belongs to $\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}$.

This $\mathbb{E}_1$-monoidal structure is inherited by the full subcategory $\mathrm{Shv}_{\mathscr{G}^{\mathrm{I}}}(\mathrm{Hec}^{\mathrm{I}}_{\mathrm{G}})_{/\mathrm{X}^{\mathrm{I}}}$: the ind-algebraic stack $\widetilde{\mathrm{Hec}}^{\mathrm{I},[2]}_{\mathrm{G}}$ is étale locally isomorphic to a product of $\mathrm{Hec}^{\mathrm{I}}_{\mathrm{G}}$ with $\mathrm{Gr}^{\mathrm{I}}_{\mathrm{G}}$ and universal local acyclicity is preserved under proper pushforward.

2.3.6. Suppose that I admits an ordered partition into nonempty finite sets $\mathrm{I} = \mathrm{I}_1 \sqcup \cdots \sqcup \mathrm{I}_k$. Let $\mathscr{F}_a$ be an object of $\mathrm{Shv}_{\mathscr{G}^{\mathrm{I}_a}}(\mathrm{Hec}^{\mathrm{I}_a}_{\mathrm{G}})$, for $1 \le a \le k$.

We may form their *external convolution product* using the morphisms in (2.5) and the isomorphism of A-gerbes (2.14):

$$\circ_{1\le a\le k}\, \mathscr{F}_a := m_!(\bigotimes_{1\le a\le k} p_a^*\mathscr{F}_a) \in \mathrm{Shv}_{\mathscr{G}^{\mathrm{I}}}(\mathrm{Hec}^{\mathrm{I}}_{\mathrm{G}}). \tag{2.18}$$

Lemma 2.3.7. *If each $\mathscr{F}_a$ belongs to $\mathrm{Sat}^{\mathrm{I}_a}_{\mathrm{G},\mathscr{G}^{\mathrm{I}_a}}$, then $\circ_{1\le a\le k}\mathscr{F}_a$ belongs to $\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}$.*

Proof. The fact that $\circ_{1\le a\le k}\mathscr{F}_a$ is universally locally acyclic relative to X^{I} is argued as in §2.3.5. It remains to show that $\circ_{1\le a\le k}\mathscr{F}_a$ is perverse over $\mathrm{Gr}^{\mathrm{I}}_{\mathrm{G}}$ relative to X^{I}.

The universal local acyclicity condition implies that this sheaf has zero vanishing cycle along any specialization of geometric points in X^{I}. Hence its geometric fibers over X^{I} are isomorphic to the nearby cycles of $\boxtimes_{1\le a\le k}\mathscr{F}_a$ over the pairwise disjoint locus in X^{I}. Finally, perversity is preserved under the nearby cycle functor [Ill94, Corollaire 4.5]. □

2.3.8. For later convenience, we define a variant of the Satake category associated to a nonempty finite set I equipped with an ordered partition $\mathrm{I} \cong \bigsqcup_{1\le a\le k}\mathrm{I}_a$ into nonempty subsets:

$$\widetilde{\mathrm{Sat}}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}} \subset \mathrm{Shv}_{\mathscr{G}^{\mathrm{I}}}(\widetilde{\mathrm{Hec}}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}_{\mathrm{G}}),$$

characterized by universal local acyclicity and perversity over $\widetilde{\mathrm{Gr}}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}_{\mathrm{G}}$. Here, the A-gerbe is the pullback of $\mathscr{G}^{\mathrm{I}}$ along the morphism m in (2.5).

Under the hypothesis of Lemma 2.3.7, the sheaf $\bigotimes_{1\le a\le k} p_a^*\mathscr{F}_a$ belongs to $\widetilde{\mathrm{Sat}}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}$, passing through the identification of A-gerbes (2.14). Indeed, this is already established in the proof of Lemma 2.3.7.

2.3.9. The triple $(\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}, \circ_{\mathrm{X}}, e_!(\underline{\mathrm{E}}))$ admits the structure of a monoidal category.

Construction. The construction (2.18) for two copies of the same nonempty finite set I produces an object $\mathscr{F}_1 \circ \mathscr{F}_2 \in \mathrm{Shv}_{\mathscr{G}^{\mathrm{I}\sqcup\mathrm{I}}}(\mathrm{Hec}^{\mathrm{I}\sqcup\mathrm{I}}_{\mathrm{G}})$. Its restriction along the diagonal $\mathrm{X}^{\mathrm{I}} \to \mathrm{X}^{\mathrm{I}\sqcup\mathrm{I}}$ is canonically identified with $\mathscr{F}_1 \circ_{\mathrm{X}} \mathscr{F}_2$.

Combining this observation with Lemma 2.3.7, we see that the $\mathbb{E}_1$-monoidal structure on $\mathrm{Shv}_{\mathscr{G}^{\mathrm{I}}}(\mathrm{Hec}^{\mathrm{I}}_{\mathrm{G}})_{/\mathrm{X}^{\mathrm{I}}}$ constructed in §2.3.5 is inherited by $\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}$. □

2.3.10. Given an unordered partition $p : \mathrm{I} \twoheadrightarrow \mathrm{J}$, i.e. $\mathrm{I} \cong \bigsqcup_{j\in\mathrm{J}}\mathrm{I}_j$ as in §2.1.3, we define a "disjoint" variant of the Satake category:

$$\mathrm{Sat}^{p}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}} \subset \mathrm{Shv}_{\mathscr{G}^{\mathrm{I}}}(\mathrm{Hec}^{\mathrm{I}}_{\mathrm{G}} \times_{\mathrm{X}} \mathrm{X}^{p}),$$

as the full subcategory consisting of objects whose pullback to $\mathrm{Gr}^{\mathrm{I}}_{\mathrm{G}} \times_{\mathrm{X}} \mathrm{X}^{p}$ belongs to $\mathrm{Perv}_{\mathscr{G}^{\mathrm{I}}}(\mathrm{Gr}^{\mathrm{I}}_{\mathrm{G}} \times_{\mathrm{X}} \mathrm{X}^{p})_{/\mathrm{X}^{p}}$.

By [HS21, Theorem 6.8], the restriction functor is fully faithful:

$$\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}} \subset \mathrm{Sat}^{p}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}, \quad \mathscr{F} \mapsto \mathscr{F}|_{\mathrm{X}^p}, \tag{2.19}$$

and its essential image is stable under subquotients.

Given $\mathscr{F}_j \in \mathrm{Sat}^{\mathrm{I}_j}_{\mathrm{G},\mathscr{G}^{\mathrm{I}_j}}$ for each $j \in \mathrm{J}$, the external tensor product $\boxtimes_{j\in\mathrm{J}} \mathscr{F}_j$ is a $(\boxtimes_{j\in\mathrm{J}} \mathscr{G}^{\mathrm{I}_j})$-twisted sheaf over $\prod_{j\in\mathrm{J}} \mathrm{Hec}^{\mathrm{I}_j}_{\mathrm{G}}$. Its restriction $\boxtimes_{j\in\mathrm{J}} \mathscr{F}_j|_{\mathrm{X}^p}$ may be viewed as a $\mathscr{G}^{\mathrm{I}}$-twisted sheaf over $\mathrm{Hec}^{\mathrm{I}}_{\mathrm{G}} \times_{\mathrm{X}} \mathrm{X}^p$ using the factorization isomorphism (2.2) and its compatibility with the A-gerbes (see §2.2.7).

Note that $\boxtimes_{j\in\mathrm{J}} \mathscr{F}_j|_{\mathrm{X}^p}$ belongs to the essential image of (2.19), and we call the corresponding object the *external fusion product*:

$$\star_{j\in\mathrm{J}} \mathscr{F}_j \in \mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}. \tag{2.20}$$

Indeed, given any ordering $\mathrm{J} \cong \{1,\cdots,k\}$, we may form $\circ_{1\le a\le k}\mathscr{F}_a \in \mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}$ using Lemma 2.3.7 whose restriction along $\mathrm{X}^p \subset \mathrm{X}^{\mathrm{I}}$ is identified with $\boxtimes_{j\in\mathrm{J}} \mathscr{F}_j|_{\mathrm{X}^p}$.

Remark 2.3.11. By this argument, any ordering on J induces an isomorphism between the external convolution product (2.18) and the external fusion product (2.20).

2.3.12. Finally, for a nonempty finite set I, the *fusion product* of two objects $\mathscr{F}_1, \mathscr{F}_2 \in \mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}$ is defined to be the restriction of $\mathscr{F}_1 \star \mathscr{F}_2 \in \mathrm{Sat}^{\mathrm{I}\sqcup\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}\sqcup\mathrm{I}}}$ along the diagonal $\mathrm{X}^{\mathrm{I}} \to \mathrm{X}^{\mathrm{I}\sqcup\mathrm{I}}$:

$$\mathscr{F}_1 \star_{\mathrm{X}^{\mathrm{I}}} \mathscr{F}_2 := (\mathscr{F}_1 \star \mathscr{F}_2)|_{\mathrm{X}^{\mathrm{I}}}. \tag{2.21}$$

The triple $(\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}, \star_{\mathrm{X}}, e_!(\underline{\mathrm{E}}))$ admits the structure of a *symmetric* monoidal category. The symmetry constraint comes from the fact that the formation of $\mathscr{F}_1 \star \mathscr{F}_2$ uses the unique unordered partition of $\{1,2\}$.

Furthermore, Remark 2.3.11 implies that the monoidal structures corresponding to $\circ_{\mathrm{X}}$ and $\star_{\mathrm{X}}$ are identified.

Remark 2.3.13. Fargues–Scholze [FS21, VI] explains another way to identify these monoidal structures, which is more natural from a higher categorical perspective.

To wit, the convolution structure upgrades $(\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}, \star_{\mathrm{X}}, e_!(\underline{\mathrm{E}}))$ to an $\mathbb{E}_1$-monoidal object in the 2-category of symmetric monoidal categories, so the two monoidal structures are identified by a variant of the Eckmann–Hilton argument.

Remark 2.3.14. Note that given a surjection of nonempty finite sets $p : \mathrm{I} \twoheadrightarrow \mathrm{J}$, with corresponding diagonal $\Delta^p : \mathrm{X}^{\mathrm{J}} \to \mathrm{X}^{\mathrm{I}}$, the restriction of $\mathscr{G}^{\mathrm{I}}$ to $\mathrm{Hec}^{\mathrm{I}}_{\mathrm{G}} \times_{\mathrm{X}^{\mathrm{I}}} \mathrm{X}^{\mathrm{J}} \cong \mathrm{Hec}^{\mathrm{J}}_{\mathrm{G}}$ is canonically isomorphic to $\mathscr{G}^{\mathrm{J}}$. These isomorphisms are furthermore compatible with compositions.

It follows that the association $(\mathrm{I} \neq \varnothing) \mapsto \mathrm{Sat}^{\mathrm{I}}_{\mathrm{H}^{\mathrm{I}},\mathrm{F}^{\mathrm{I}}}$ defines a functor from the category of nonempty finite sets with surjections to the 2-category of tensor categories.

2.4. The equivalence.

2.4.1. We now assume that $\mathrm{G} \to \mathrm{X}$ is a reductive group scheme. Notations for the based root data of G are as in §1.6.2.

Recall that the algebraic fundamental group of G is the sheaf of abelian groups $\pi_1(\mathrm{G}) := \Lambda/\Lambda^r$, where Λ^r is the span of Φ.

Let $2\check{\rho} \in \check{\Lambda}$ denote the sum of positive roots. Its reduction mod 2 defines a character of $\pi_1(\mathrm{G})$ valued in $\mathbb{Z}/2$.

In order to state a properly normalized version of the Satake equivalence, we shall assume the existence of and fix a square root $\underline{\mathrm{E}}(\frac{1}{2})$ of $\underline{\mathrm{E}}$.

2.4.2. For a nonempty finite set I, we tweak the commutativity constraint of $\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}$ in the usual way.

Indeed, the connected components of $\mathrm{Gr}^{\mathrm{I}}_{\mathrm{G}}$ are indexed by $\pi_1(\mathrm{G})$. The symmetric monoidal category $\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}$ acquires a decomposition according to the support of its object:

$$\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}} \cong \bigoplus_{\lambda\in\pi_1(\mathrm{G})} (\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}})^{\lambda}. \tag{2.22}$$

This decomposition is compatible with the monoidal structure.

Set ${}^{+}\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}} := \mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}$ as a *monoidal* category, but whose commutativity constraint for a pair of objects from $(\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}})^{\lambda_1}$, $(\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}})^{\lambda_2}$ is multiplied by $(-1)^{\langle 2\check{\rho},\lambda_1\rangle\langle 2\check{\rho},\lambda_2\rangle}$.

2.4.3. Let (H,F) be the metaplectic dual data associated to (G,μ) as in §1.6. Recall the twist F_{ϑ} of F by the ϑ-characteristic of X (see §1.6.16).

Applying the construction of §1.7.6 to the pair $(\mathrm{H},\mathrm{F}_{\vartheta})$, we obtain a tensor category $\mathrm{Rep}^{\mathrm{I}}_{\mathrm{H}^{\mathrm{I}},\mathrm{F}^{\mathrm{I}}_{\vartheta}}$ for each nonempty finite set I.

Theorem 2.4.4 (Geometric Satake equivalence)**.** *For each nonempty finite set* I*, there is a canonical equivalence of tensor categories:*

$$ {}^{+}\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}} \cong \mathrm{Rep}^{\mathrm{I}}_{\mathrm{H}^{\mathrm{I}},\mathrm{F}^{\mathrm{I}}_{\vartheta}}. \tag{2.23}$$

2.4.5. The equivalence (2.23) which we shall construct comes equipped with two additional pieces of compatibility data.

First, it is compatible with restrictions along the diagonals. Namely, the following diagram is canonically commutative for every surjection of nonempty finite sets $p:\mathrm{I}\twoheadrightarrow\mathrm{J}$:

$$\begin{array}{ccc} {}^{+}\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}} & \cong & \mathrm{Rep}^{\mathrm{I}}_{\mathrm{H}^{\mathrm{I}},\mathrm{F}^{\mathrm{I}}_{\vartheta}} \\ \downarrow{\scriptstyle (\Delta^p)^*} & & \downarrow{\scriptstyle (1.33)} \\ {}^{+}\mathrm{Sat}^{\mathrm{J}}_{\mathrm{G},\mathscr{G}^{\mathrm{J}}} & \cong & \mathrm{Rep}^{\mathrm{J}}_{\mathrm{H}^{\mathrm{J}},\mathrm{F}^{\mathrm{J}}_{\vartheta}} \end{array} \tag{2.24}$$

where $\Delta^p:\mathrm{X}^{\mathrm{J}}\to\mathrm{X}^{\mathrm{I}}$ denotes the corresponding diagonal. The commutativity witness of (2.24) is compatible with compositions.

Secondly, external fusion product (2.20) of the Satake category corresponds to the external tensor product of representations. Namely, writing $\mathrm{I}_j := p^{-1}(j)$, the following diagram is canonically commutative:

$$\begin{array}{ccc} \prod_{j\in\mathrm{J}} {}^{+}\mathrm{Sat}^{\mathrm{I}_j}_{\mathrm{G},\mathscr{G}^{\mathrm{I}_j}} & \cong & \prod_{j\in\mathrm{J}} \mathrm{Rep}^{\mathrm{I}_j}_{\mathrm{H}^{\mathrm{I}_j},\mathrm{F}^{\mathrm{I}_j}_{\vartheta}} \\ \downarrow{\scriptstyle \star} & & \downarrow{\scriptstyle \boxtimes} \\ {}^{+}\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}} & \cong & \mathrm{Rep}^{\mathrm{I}}_{\mathrm{H}^{\mathrm{I}},\mathrm{F}^{\mathrm{I}}_{\vartheta}} \end{array} \tag{2.25}$$

compatibly with compositions.

Remark 2.4.6. The equivalence (2.23) is of étale local nature over X. In particular, it may be viewed as an equivalence of stacks of tensor categories on the étale site of X.

2.4.7. Let us specialize to the case $\mathrm{I}=\{1\}$. Viewing (2.23) as an equivalence of stacks of tensor categories on $\mathrm{X}_{\text{ét}}$, its stalk at a closed point $x\in\mathrm{X}$ yields an equivalence of tensor categories:

$$ {}^{+}\mathrm{Sat}_{\mathrm{G},\mathscr{G},x} \cong \mathrm{Rep}_{\mathrm{H},\mathrm{F}_{\vartheta},x}. \tag{2.26}$$

Choosing an algebraic closure $k_x \subset \bar{k}_x$ (corresponding to a morphism $\bar{x} \to x$), the right-hand-side of (2.26) is *monoidally* equivalent to $\mathrm{Rep}_{\mathrm{H},{}^0\mathrm{F}_\vartheta,x}$, hence to the representation category $\mathrm{Rep}^{\mathrm{alg}}({}^{\mathrm{L}}\mathrm{H}_{x,\vartheta})$ according to Remark 1.7.5. (To define ${}^{\mathrm{L}}\mathrm{H}_{x,\vartheta}$, one may need to replace E by a finite extension.)

If X satisfies condition (1.1), the local (integral) L-group ${}^{\mathrm{L}}\mathrm{H}_{x,\vartheta}$ is also identified with the restriction of ${}^{\mathrm{L}}\mathrm{H}_{\mathrm{X},\vartheta}$ along $\pi_1(x,\bar{x}) \to \pi_1(\mathrm{X},\bar{x})$.

2.4.8. Suppose that k is a finite field. We obtain a homomorphism of E-algebras:

$$\begin{aligned}\mathrm{K}_0(\mathrm{Rep}^{\mathrm{alg}}({}^{\mathrm{L}}\mathrm{H}_{x,\vartheta})) \otimes \mathrm{E} &\cong \mathrm{K}_0({}^{+}\mathrm{Sat}_{\mathrm{G},\mathscr{G},x}) \otimes \mathrm{E} \\ &\to \mathrm{Fun}_c(\mathrm{G}(\mathscr{O}_x)\backslash\widetilde{\mathrm{G}}_x/\mathrm{G}(\mathscr{O}_x), \mathrm{A} \subset \mathrm{E}^\times),\end{aligned} \tag{2.27}$$

where the first isomorphism is the application of the *monoidal* invariant $\mathrm{K}_0(-)\otimes\mathrm{E}$ to (2.26), and the second homomorphism is the trace of the geometric Frobenius $\varphi_x \in \mathrm{Gal}(\bar{k}_x/k_x)$ at each k_x-point of $\mathrm{Hec}_{\mathrm{G},x}$.

For an object $\mathrm{V} \in \mathrm{Rep}^{\mathrm{alg}}({}^{\mathrm{L}}\mathrm{H}_{x,\vartheta})$, we denote by $h_{\mathrm{V},x}$ its image under (2.27), and call it the (unramified) *Hecke operator* associated to V at the point x.

Remark 2.4.9. Let $\mathrm{H}_{\varphi_x} \subset {}^{\mathrm{L}}\mathrm{H}_{x,\vartheta}$ denote the preimage of φ_x. Using the argument of [Zhu17, §5.6], it is possible to show that (2.27) factors through an isomorphism of E-algebras:

$$\Gamma(\mathrm{H}_{\varphi_x}/\!/\mathrm{H}, \mathscr{O}) \cong \mathrm{Fun}_c(\mathrm{G}(\mathscr{O}_x)\backslash\widetilde{\mathrm{G}}_x/\mathrm{G}(\mathscr{O}_x), \mathrm{A} \subset \mathrm{E}^\times) \tag{2.28}$$

where the left-hand-side is the E-algebra of algebraic functions on H_{φ_x} invariant under H-conjugation. It receives a morphism from $\mathrm{K}_0(\mathrm{Rep}^{\mathrm{alg}}({}^{\mathrm{L}}\mathrm{H}_{x,\vartheta})) \otimes \mathrm{E}$ by mapping V to the character of its restriction to H_{φ_x}.

The isomorphism (2.28) for metaplectic covers arising from algebraic K-theory over any local field has already been obtained by McNamara [McN12, Theorem 10.1]. (We will not use (2.28) in this article.)

2.5. The Satake functors.

2.5.1. We keep the notations of §2.4.

2.5.2. For a nonempty finite set I, we replace ${}^{+}\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}$ by $\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}$ on one side of the geometric Satake equivalence and F_ϑ by its $\mathbb{Z}$-linear component ${}^0\mathrm{F}_\vartheta$ (see §1.6) in the formation of the other side. Both modifications only change the commutativity constraint.

In this manner, we obtain from (2.23) an equivalence of *monoidal* categories:

$$\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}} \cong \mathrm{Rep}^{\mathrm{I}}_{\mathrm{H}^{\mathrm{I}},{}^0\mathrm{F}^{\mathrm{I}}_\vartheta}. \tag{2.29}$$

For a surjection of nonempty finite sets $p : \mathrm{I} \twoheadrightarrow \mathrm{J}$, the compatibility data of (2.29) with respect to restrictions along the diagonal and external fusion products are given by (2.24) and (2.25) on the underlying monoidal categories.

2.5.3. Let us now assume that X is connected and satisfies condition (1.1).

We fix a geometric point $\bar{\eta}$ of X. Denote by ${}^0\mathrm{F}$ (resp. ${}^0\mathrm{F}_\vartheta$) the $\mathbb{Z}$-linear component of F constructed as in §1.6. We fix a rigidification of ${}^0\mathrm{F}_\vartheta$ along $\bar{\eta}$.

In this set-up, we have the extension (1.28):

$$1 \to \mathrm{H}_{\bar{\eta}}(\mathrm{E}) \to {}^{\mathrm{L}}\mathrm{H}_{\mathrm{X},\vartheta} \to \pi_1(\mathrm{X},\bar{\eta}) \to 1,$$

where $\mathrm{H}_{\bar{\eta}}$ is the fiber of H at $\bar{\eta}$, viewed as a pinned split reductive group over E.

2.5.4. Let I be a nonempty finite set together with an ordered partition $\mathrm{I} \cong \bigsqcup_{1\le a\le k} \mathrm{I}_a$ into nonempty finite subsets.

Composition of the equivalence (1.31), the external tensor product, and (2.29) defines a functor of monoidal categories:

$$\begin{aligned}\prod_{i\in \mathrm{I}} \mathrm{Rep}^{\mathrm{alg}}({}^{\mathrm{L}}\mathrm{H}_{\mathrm{X},\vartheta}) \cong \prod_{i\in \mathrm{I}} \mathrm{Rep}^{\{i\}}_{\mathrm{H},{}^0\mathrm{F}_\vartheta} &\xrightarrow{\boxtimes} \prod_{1\le a\le k} \mathrm{Rep}^{\mathrm{I}_a}_{\mathrm{H}^{\mathrm{I}_a},{}^0\mathrm{F}^{\mathrm{I}_a}_\vartheta} \\ &\cong \prod_{1\le a\le k} \mathrm{Sat}^{\mathrm{I}_a}_{\mathrm{G},\mathscr{G}^{\mathrm{I}_a}} \xrightarrow{\otimes p_a^*} \widetilde{\mathrm{Sat}}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}},\end{aligned} \tag{2.30}$$

where the functors p_a^* are pullbacks along the morphism in (2.5) (see §2.3.8).

We shall argue that (2.30) factors through the category $\mathrm{Rep}^{\mathrm{alg}}(({}^{\mathrm{L}}\mathrm{H}_{\mathrm{X},\vartheta})^{\mathrm{I}})$. This can be done by playing with the regular representation as in [Gai07, Appendix B], but we supply another argument.

2.5.5. Let Γ be a profinite group and ${}^{\mathrm{L}}\mathrm{H}$ an extension of Γ by $\mathrm{H}_{\bar\eta}(\mathrm{E})$. For a nonempty finite set I, the I-fold product $({}^{\mathrm{L}}\mathrm{H})^{\mathrm{I}}$ is an extension of Γ^{I} by $\mathrm{H}^{\mathrm{I}}_{\bar\eta}(\mathrm{E})$. In particular, we may form the tensor category $\mathrm{Rep}^{\mathrm{alg}}(({}^{\mathrm{L}}\mathrm{H})^{\mathrm{I}})$ (see Remark 1.7.5).

Recall the notion of tensor product of E-linear abelian categories in [Del90, 5.1]: it is the universal recipient of E-multilinear functors which are right exact in each factor.

Over a perfect field, the tensor product of tensor categories satisfying a finiteness condition acquires a canonical tensor structure ([Del90, 5.17]). This applies to $\mathrm{Rep}^{\mathrm{alg}}(({}^{\mathrm{L}}\mathrm{H}))$.

Lemma 2.5.6. *The tensor product of restrictions of representations along each projection* $({}^{\mathrm{L}}\mathrm{H})^{\mathrm{I}} \to {}^{\mathrm{L}}\mathrm{H}$ *induces an equivalence of tensor categories:*

$$\bigotimes_{i\in \mathrm{I}} \mathrm{Rep}^{\mathrm{alg}}({}^{\mathrm{L}}\mathrm{H}) \cong \mathrm{Rep}^{\mathrm{alg}}(({}^{\mathrm{L}}\mathrm{H})^{\mathrm{I}}). \tag{2.31}$$

Proof. The tensor category $\mathrm{Rep}^{\mathrm{alg}}({}^{\mathrm{L}}\mathrm{H})$ is Tannakian with fiber functor ω being the functor of forgetting the ${}^{\mathrm{L}}\mathrm{H}$-action. The sheaf of automorphisms $\underline{\mathrm{Aut}}(\omega)$ is thus representable by an affine group scheme ${}^{\mathrm{L}}\mathrm{H}^{\mathrm{alg}}$ over E, and $\mathrm{Rep}^{\mathrm{alg}}({}^{\mathrm{L}}\mathrm{H})$ is equivalent to the category of finite-dimensional representations of ${}^{\mathrm{L}}\mathrm{H}^{\mathrm{alg}}$.

Given a finite-dimensional E-vector space V, its lift to an object of $\mathrm{Rep}^{\mathrm{alg}}(({}^{\mathrm{L}}\mathrm{H})^{\mathrm{I}})$ is equivalent to commuting continuous actions of ${}^{\mathrm{L}}\mathrm{H}$ indexed by I, satisfying the algebraicity condition over $\mathrm{H}(\mathrm{E})$: this is equivalent to an action of $({}^{\mathrm{L}}\mathrm{H}^{\mathrm{alg}})^{\mathrm{I}}$. Hence we have $(({}^{\mathrm{L}}\mathrm{H})^{\mathrm{I}})^{\mathrm{alg}} \cong ({}^{\mathrm{L}}\mathrm{H}^{\mathrm{alg}})^{\mathrm{I}}$ and the equivalence (2.31) follows from [Del90, 6.21]. ☐

2.5.7. The functor (2.30), being E-multilinear and right exact in each factor, canonically factors through a right exact monoidal functor according to Lemma 2.5.6:

$$\mathscr{S}^{\mathrm{I}_1,\cdots,\mathrm{I}_k} : \mathrm{Rep}^{\mathrm{alg}}(({}^{\mathrm{L}}\mathrm{H}_{\mathrm{X},\vartheta})^{\mathrm{I}}) \to \widetilde{\mathrm{Sat}}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}. \tag{2.32}$$

We shall call $\mathscr{S}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}$ the *Satake functor* associated to the nonempty finite set I together with the ordered partition $\mathrm{I} \cong \bigsqcup_{1\le a\le k} \mathrm{I}_a$.

Furthermore, the functors (2.32) are compatible with change of the partitioned set. More precisely, given nonempty finite sets I, J, nonempty ordered finite sets K_1, K_2 and a commutative diagram of surjective morphisms where q is order-preserving:

$$\begin{array}{ccccc} \mathrm{I} & \twoheadrightarrow & \mathrm{K}_1 & \cong & \{1,\cdots,k_1\} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} & & \\ \mathrm{J} & \twoheadrightarrow & \mathrm{K}_2 & \cong & \{1,\cdots,k_2\} \end{array} \tag{2.33}$$

we have a canonically commutative diagram of E-linear abelian categories:

$$\begin{array}{ccc} \mathrm{Rep}^{\mathrm{alg}}((^{\mathrm{L}}\mathrm{H}_{\mathrm{X},\vartheta})^{\mathrm{I}}) & \xrightarrow{\mathscr{S}^{\mathrm{I}_1,\cdots,\mathrm{I}_{k_1}}} & \widetilde{\mathrm{Sat}}^{\mathrm{I}_1,\cdots,\mathrm{I}_{k_1}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}} \\ \downarrow{\scriptstyle (\Delta^p)^*} & & \downarrow{\scriptstyle (m^q)_! \cdot (\Delta^p)^*} \\ \mathrm{Rep}^{\mathrm{alg}}((^{\mathrm{L}}\mathrm{H}_{\mathrm{X},\vartheta})^{\mathrm{J}}) & \xrightarrow{\mathscr{S}^{\mathrm{J}_1,\cdots,\mathrm{J}_{k_2}}} & \widetilde{\mathrm{Sat}}^{\mathrm{J}_1,\cdots,\mathrm{J}_{k_2}}_{\mathrm{G},\mathscr{G}^{\mathrm{J}}} \end{array} \tag{2.34}$$

Here, $m^q : \widetilde{\mathrm{Hec}}_{\mathrm{G}}^{\mathrm{I}_1,\cdots,\mathrm{I}_{k_1}} \times_{\mathrm{X}^{\mathrm{I}}} \mathrm{X}^{\mathrm{J}} \to \widetilde{\mathrm{Hec}}_{\mathrm{G}}^{\mathrm{J}_1,\cdots,\mathrm{J}_{k_2}}$ is defined by composing the modifications corresponding to those segments in K_1 defined by fibers of q.

The commutativity of (2.34) follows from a combination of (2.24), (2.25), and the identification of external fusion and convolution products (Remark 2.3.11). It is compatible with compositions of squares of the type (2.33).

Remark 2.5.8. Our Satake functors and their compatibility data (2.34) are parallel to the assertions of [Laf18, Théorème 1.17].

Indeed, (c) of *loc.cit.* comes from our definition of the Satake functors. Assertions (b) and (d) are obtained by setting I = J, respectively $\mathrm{K}_1 = \mathrm{K}_2$ in (2.34). (However, we only use functoriality with respect to surjective maps.) The remaining assertion (a) will be established in the course of the proof of Theorem 2.4.4.

3. Proofs

This section is entirely dedicated to the proof of Theorem 2.4.4. The proof follows the same overarching structure as [MV07] and shares many common features with [FL10].

The first two subsections §3.1–3.2 treat the case for split tori. The difficulties in this case are mostly of categorical nature, and closely related statements have appeared in [Rei12] and [GL18]. However, since our metaplectic dual data are *not* defined using factorization gerbes, we are obligated to supply new proofs.

Then, we study $\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}$ for $\mathrm{I} = \{1\}$ as an abelian category in §3.3. The methods in the non-metaplectic setting carry over with minimal modifications.

Subsections §3.4–3.6 are dedicated to the study of the constant term functor. The construction of the "fiber functor" in the metaplectic context differs substantially from the non-metaplectic one. It requires constructing a tensor decomposition of $\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}$ according to $\mathrm{Z}^{\mathrm{I}}_{\mathrm{H}}$-weights. The method we use involves a study of the A-gerbe $\mathscr{G}^{\mathrm{I}}$ over the Mirković-Vilonen cycles, which is hopefully interesting in its own right.

The last subsection §3.7 reconstructs H using a relative Tannakian formalism. It is a variant of [FS21, VI.10] and contains few surprises.

3.0.1. We remain in the context of §2.0.1 throughout this section. Furthermore, we assume the existence of and fix a square root $\underline{\mathrm{E}}(\frac{1}{2})$ of $\underline{\mathrm{E}}$. (It will be used from §3.3 onwards.)

For a nonempty finite set I, the notation $\mathscr{G}^{\mathrm{I}}$ stands for the A-gerbe on $\mathrm{Hec}^{\mathrm{I}}_{\mathrm{G}}$ constructed in §2.2. The metaplectic dual data (H, F) are as in §1.6.

3.1. Split tori: reduction to $\mathrm{T}^\sharp$.

3.1.1. Suppose that G = T is a split torus with sheaf of cocharacters Λ.

Recall from §1.6.5 that μ defines a quadratic form Q on Λ, and we let $\Lambda^\sharp \subset \Lambda$ be the kernel of the associated symmetric form b. Let $\mathrm{T}^\sharp \to \mathrm{T}$ denote the corresponding isogeny of split tori.

For a nonempty finite set I, the restriction of $\mathscr{G}^{\mathrm{I}}$ along $\mathrm{Hec}^{\mathrm{I}}_{\mathrm{T}^\sharp} \to \mathrm{Hec}^{\mathrm{I}}_{\mathrm{T}}$ is identified with the A-gerbe associated to the restriction of μ to $\mathrm{T}^\sharp$, by functoriality of the construction. The notation $\mathscr{G}^{\mathrm{I}}$ is retained for its restriction.

The goal of this subsection is to prove the following statement.

Proposition 3.1.2. *Let* I *be a nonempty finite set. Pushing forward along* $\mathrm{T}^\sharp \to \mathrm{T}$ *defines an equivalence of tensor categories:*

$$\mathrm{Sat}^{\mathrm{I}}_{\mathrm{T}^\sharp, \mathscr{G}^{\mathrm{I}}} \cong \mathrm{Sat}^{\mathrm{I}}_{\mathrm{T}, \mathscr{G}^{\mathrm{I}}}. \tag{3.1}$$

3.1.3. For a tuple $\lambda^{\mathrm{I}} = (\lambda^i)_{i\in \mathrm{I}}$ of elements of Λ, we have a closed immersion $\mathrm{X}^{\mathrm{I}} \to \mathrm{Gr}^{\mathrm{I}}_{\mathrm{T}}$ sending an S-point x^{I} to the T-torsor $\mathscr{O}(\sum_{i\in\mathrm{I}} \lambda^i x^i)$ together with its canonical trivialization off $\Gamma_{x^{\mathrm{I}}}$. We view its image as a closed subscheme:

$$\mathrm{X}^{\lambda^{\mathrm{I}}} \subset \mathrm{Gr}^{\mathrm{I}}_{\mathrm{T}}. \tag{3.2}$$

Since the $\mathrm{L}^{\mathrm{I}}_+(\mathrm{T})$-action on $\mathrm{Gr}^{\mathrm{I}}_{\mathrm{T}}$ is trivial, we obtain a closed substack $\mathrm{B}_{\mathrm{X}^{\lambda^{\mathrm{I}}}}(\mathrm{L}^{\mathrm{I}}_+(\mathrm{T})) \subset \mathrm{Hec}^{\mathrm{I}}_{\mathrm{T}}$ by taking the quotient of (3.2) by the $\mathrm{L}^{\mathrm{I}}_+(\mathrm{T})$-action.

Consider the special case $\mathrm{I} = \{1\}$ and $\lambda^1 = \lambda \in \Lambda$. Restricting sections of T along $\Gamma_x \subset \mathrm{D}_x$ defines a morphism $\mathrm{L}^{\{1\}}_+(\mathrm{T}) \to \mathrm{T}$. We label the corresponding morphisms on their classifying stacks (relative to $\mathrm{X}^\lambda \cong \mathrm{X}$) as in the following diagram:

$$\begin{array}{ccccc} & & \mathrm{X}^\lambda & \cong & \mathrm{X}^\lambda \\ & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle \cong} \\ \mathrm{Hec}^{\{1\}}_{\mathrm{T}} & \xleftarrow{i^\lambda} & \mathrm{B}_{\mathrm{X}^\lambda}(\mathrm{L}^{\{1\}}_+\mathrm{T}) & \xrightarrow{p} & \mathrm{X} \\ & & \downarrow{\scriptstyle p_{\mathrm{T}}} & & \downarrow{\scriptstyle \cong} \\ & & \mathrm{B}_{\mathrm{X}}(\mathrm{T}) & \longrightarrow & \mathrm{X} \end{array} \tag{3.3}$$

3.1.4. Recall from §1.3.6 that any section $a \in \underline{\mathrm{A}}(-1)$ defines the multiplicative A-torsor Ψ^a on $\mathbb{G}_m$, thus a rigidified section of $\mathrm{B}^2\underline{\mathrm{A}}$ over $\mathrm{B}\mathbb{G}_m$.

There is an isomorphism of étale sheaves over X:

$$\check{\Lambda} \otimes \mathrm{A}(-1) \cong \underline{\mathrm{Maps}}_e(\mathrm{B}_{\mathrm{X}}\mathrm{T}, \mathrm{B}^2_{\mathrm{X}}\underline{\mathrm{A}}), \quad x \otimes a \mapsto x^*(\Psi^a), \tag{3.4}$$

where the target is the sheaf of rigidified sections of $\mathrm{B}^2(\underline{\mathrm{A}})$ over $\mathrm{B}_{\mathrm{X}}(\mathrm{T})$. This follows from the calculation of étale cohomlogy of $\mathrm{B}_{\mathrm{X}}(\mathrm{T})$ as in [Zha22, §4].

In particular, the character $b(-,\lambda) : \Lambda \to \underline{\mathrm{A}}(-1)$ defines a rigidified section of $\mathrm{B}^2(\underline{\mathrm{A}})$ over $\mathrm{B}_{\mathrm{X}}(\mathrm{T})$, to be denoted by $\Psi^{b(-,\lambda)}$.

3.1.5. Next, we recall from [Zha22, §4.5] that the $\mathbb{E}_1$-*monoidal* morphism underlying $\mathrm{F} : \Lambda^\sharp \to \mathrm{B}^2_{\mathrm{X}}(\underline{\mathrm{A}})$ extends to Λ.

Let us reformulate the construction of *op.cit.* to bring out its parallel with §2.2.2.

Indeed, consider the affine line $\mathbb{A}^1_{\mathrm{X}} \to \mathrm{X}$ and the closed and open immersions corresponding to the zero section: $i : \mathrm{X} \to \mathbb{A}^1_{\mathrm{X}}$, $j : \mathbb{G}_{m,\mathrm{X}} \to \mathbb{A}^1_{\mathrm{X}}$.

Composition with the $\mathbb{E}_1$-monoidal morphism $\mathrm{T} \to \mathrm{B}^3_{\mathrm{X}}\underline{\mathrm{A}}(1)$ corresponding to μ defines a morphism of étale sheaves of $\mathbb{E}_1$-monoids over X:

$$\mu_* : \underline{\Gamma}(\mathbb{G}_{m,\mathrm{X}}, \mathrm{T}) \to \underline{\Gamma}(\mathbb{G}_{m,\mathrm{X}}, \mathrm{B}^3_{\mathrm{X}}\underline{\mathrm{A}}(1)). \tag{3.5}$$

The target of (3.5) is the sheaf of ∞-groupoids underlying $j_*\underline{\mathrm{A}}(1)[3]$. It admits a morphism to $i^!\underline{\mathrm{A}}(1)[4]$ via the Cousin triangle, while the latter complex is isomorphic to $\underline{\mathrm{A}}[2]$ by purity.

The desired $\mathbb{E}_1$-monoidal extension of F is defined as the composition:

$$\begin{aligned} \mathrm{F} : \Lambda &\to \underline{\Gamma}(\mathbb{G}_{m,\mathrm{X}}, \mathrm{T}) \\ &\xrightarrow{\mu_*} \underline{\Gamma}(\mathbb{G}_{m,\mathrm{X}}, \mathrm{B}^3_{\mathrm{X}}\underline{\mathrm{A}}(1)) \to \mathrm{B}^2_{\mathrm{X}}(\underline{\mathrm{A}}), \end{aligned} \tag{3.6}$$

where the last map is the one explained above.

In particular, for the curve $\mathrm{X} = \mathbb{A}^1$, (3.6) is nothing but the pullback of the $\mathbb{E}_1$-monoidal gerbe $\mathscr{G}^{\{1\}} : \mathrm{L}^{\{1\}}(\mathrm{T}) \to \mathrm{B}^2_{\mathrm{X}}(\underline{\mathrm{A}})$ along the natural map $\Lambda \to \mathrm{L}^{\{1\}}(\mathrm{T})$.

3.1.6. For each $\lambda \in \Lambda$, we denote by $\mathrm{F}_\vartheta(\lambda)$ the A-gerbe $\mathrm{F}(\lambda) \otimes \omega_{\mathrm{X}}^{\mathrm{Q}(\lambda)}$, where $\omega_{\mathrm{X}}^{\mathrm{Q}(\lambda)}$ is the A-gerbe induced from ω_{X} along $\Psi^{\mathrm{Q}(\lambda)} : \mathbb{G}_m \to \mathrm{B}(\mathrm{A})$. The association $\lambda \mapsto \mathrm{F}_\vartheta(\lambda)$ extends the $\mathbb{E}_\infty$-monoidal morphism F_ϑ in §1.6.16 as a *pointed* morphism.

In the following calculation, we show that the relationship between $\mathscr{G}^{\{1\}}$ and F holds over a general smooth curve X, up to modification by ω_{X}.

Lemma 3.1.7. *For each $\lambda \in \Lambda$, there is a canonical isomorphism of* A*-gerbes, in reference to the morphisms in* (3.3)*:*

$$(i^\lambda)^* \mathscr{G}^{\{1\}} \cong p^* \mathrm{F}_\vartheta(\lambda) \otimes (p_{\mathrm{T}})^* (\Psi^{b(-,\lambda)}). \tag{3.7}$$

(For $\lambda = 0$, this is the identity automorphism of the trivial A*-gerbe.)*

3.1.8. We begin the proof of Lemma 3.1.7 with an observation: since the kernel of the projection $\mathrm{L}^{\mathrm{I}}_+(\mathrm{T}) \to \mathrm{T}$ is pro-unipotent, pulling back along p_{T} defines an equivalence on the (resp. discrete) groupoid of (resp. rigidified) A-gerbes.

It thus suffices to perform the tasks below:

(1) construct a canonical isomorphism:

$$e^* (i^\lambda)^* \mathscr{G}^{\{1\}} \cong \mathrm{F}_\vartheta(\lambda); \tag{3.8}$$

(2) show that the rigidified A-gerbe $(i^\lambda)^* \mathscr{G}^{\{1\}} \otimes p^* \mathrm{F}_\vartheta(\lambda)^{\otimes -1}$ equals the character $b(-,\lambda)$ under the isomorphism (3.4).

Construction of (3.8). Pulling back along the cocharacter $\mathbb{G}_m \to \mathrm{T}$ defined by λ, we reduce to the case $\mathrm{T} = \mathbb{G}_m$ and $\lambda = 1$.

Let $\Delta : \mathrm{X} \to \mathrm{X}^2$ denote the diagonal immersion. Viewing $\mathscr{O}(\Delta)$ as a section of $\Delta^!(\mathbb{G}_m)[1]$, we may compose it with the pullback of $\mu : \mathrm{B}_{\mathrm{X}}(\mathbb{G}_m) \to \mathrm{B}^4_{\mathrm{X}}\underline{\mathrm{A}}(1)$ along the second projection $p_2 : \mathrm{X}^2 \to \mathrm{X}$ to obtain a section $\mu_* \mathscr{O}(\Delta)$ of the étale sheaf $\Delta^! \underline{\mathrm{A}}(1)[4]$.

The construction of $\mathscr{G}^{\{1\}}$ in §2.2 shows that the restriction $e^*(i^\lambda)^* \mathscr{G}^{\{1\}}$ is identified with the image of $\mu_* \mathscr{O}(\Delta)$ under the purity isomorphism:

$$\Gamma(\mathrm{X}, \Delta^! \underline{\mathrm{A}}(1)[4]) \cong \Gamma(\mathrm{X}, \underline{\mathrm{A}}[2]). \tag{3.9}$$

On the other hand, recall that étale metaplectic covers of $\mathbb{G}_m$ are parametrized by a direct product of étale sheaves over X of 2-groupoids ([Zha22, §4.3]):

$$\underline{\mathrm{Maps}}_e(\mathrm{B}_{\mathrm{X}}\mathbb{G}_m, \mathrm{B}^4_{\mathrm{X}}\underline{\mathrm{A}}(1)) \cong \mathrm{B}^2_{\mathrm{X}}(\underline{\mathrm{A}}) \times \underline{\mathrm{A}}(-1).$$

Hence it suffices to construct (3.8) for μ belonging to either of the factors.

Case: $\mathrm{B}^2_{\mathrm{X}}(\underline{\mathrm{A}})$. This μ is a $\mathbb{Z}$-linear morphism $\mathrm{B}_{\mathrm{X}}\mathbb{G}_m \to \mathrm{B}^4_{\mathrm{X}}\underline{\mathrm{A}}(1)$, or equivalently a morphism of complexes $\mathbb{G}_m[1] \to \underline{\mathrm{A}}(1)[4]$ over X, or yet equivalently a morphism of complexes $\lim_n(\mathbb{Z}/n) \to \underline{\mathrm{A}}[2]$ (over $n \geq 0$ and $p \nmid n$).

In this case, $\mathrm{Q} = 0$ and $\mathrm{F}_\vartheta(1) \cong \mathrm{F}(1)$ is the section of $\underline{\mathrm{A}}[2]$ corresponding to the value of μ at $1 \in \lim_n(\mathbb{Z}/n)$.

Naturality of the purity isomorphism (3.9) with respect to the change of coefficients $p_2^*(\mu) : \lim_n(\mathbb{Z}/n) \to \underline{\mathrm{A}}[2]$ over X^2 gives rise to a commutative diagram:

$$\begin{array}{ccc} \lim_n \Delta^!(\mu_n)[2] & \xrightarrow{p_2^*(\mu)} & \Delta^!\underline{\mathrm{A}}(1)[4] \\ \downarrow\cong & & \downarrow\cong \\ \lim_n(\mathbb{Z}/n) & \xrightarrow{\mu} & \underline{\mathrm{A}}[2] \end{array}$$

The image of $\Psi_*\mathscr{O}(\Delta)$ under the upper circuit yields $e^*(i^1)^*\mathscr{G}^{\{1\}}$, whereas its image under the lower circuit yields $\mathrm{F}_\vartheta(1)$.

Case: $\mathrm{A}(-1)$. This μ is of the form $(\Psi \cup \Psi)^a := a_*(\Psi \cup \Psi)$ for a section $a \in \underline{\mathrm{A}}(-1)$. Here, Ψ is viewed as a rigidified section of $\lim_n(\mathrm{B}^2\mu_n)$ over $\mathrm{B}_\mathrm{X}\mathbb{G}_m$.

In this case, $\mathrm{Q}(1) = a$ and $\mathrm{F}(1)$ is trivialized, so $\mathrm{F}_\vartheta(1) \cong \omega_\mathrm{X}^a$ in the notation of §1.6.15.

Naturality of the purity isomorphism with respect to the change of coefficients $\mathscr{O}(\Delta)^a : \lim_n(\mathbb{Z}/n) \to \underline{\mathrm{A}}[2]$ gives rise to a commutative diagram:

$$\begin{array}{ccc} \lim_n \Delta^!(\mu_n)[2] & \xrightarrow{\mathscr{O}(\Delta)^a} & \Delta^!\underline{\mathrm{A}}(1)[4] \\ \downarrow\cong & & \downarrow\cong \\ \lim_n(\mathbb{Z}/n) & \xrightarrow{\omega_\mathrm{X}^a} & \underline{\mathrm{A}}[2] \end{array}$$

where ω_X^a arises from the isomorphism $\Delta^*\mathscr{O}(\Delta) \cong \omega_\mathrm{X}$.

The image of $\Psi_*\mathscr{O}(\Delta)$ under the upper circuit yields $e^*(i^1)^*\mathscr{G}^{\{1\}}$, by identifying the self-cup product $(\Psi_*\mathscr{O}(\Delta) \cup \Psi_*\mathscr{O}(\Delta))^a$ with the Yoneda product, namely the section induced from $\Psi_*\mathscr{O}(\Delta)$ along $\mathscr{O}(\Delta)^a$. Its image under the lower circuit yields $\mathrm{F}_\vartheta(1)$. □

Proof of Lemma 3.1.7. Only task (2) of §3.1.8 remains. Since it asserts the equality of two sections of $\check{\Lambda} \otimes \underline{\mathrm{A}}(-1)$ over X^λ, we may verify it over an arbitrary geometric point $\bar{x}$ of X.

We omit the superscript $\{1\}$ and use the subscript "$\bar{x}$" to mean the base change to $\bar{x}$. The base change of X^λ to $\bar{x}$ will simply be written as $\bar{x}^\lambda$.

We fix a uniformizer ϖ at $\bar{x}$, which defines a geometric point ϖ^λ of $\mathrm{L}(\mathrm{T})_{\bar{x}}$.

By construction, the A-gerbe over $(\mathrm{Hec}_\mathrm{T})_{\bar{x}}$ is defined by an $\mathbb{E}_1$-monoidal A-gerbe:

$$\mathscr{G} : \mathrm{L}(\mathrm{T})_{\bar{x}} \to \mathrm{B}^2(\underline{\mathrm{A}}) \tag{3.10}$$

along with its trivialization over $\mathrm{L}_+(\mathrm{T})_{\bar{x}}$ as in (2.8). Indeed, the trivialization equips $\mathscr{G}$ with left and right $\mathrm{L}_+(\mathrm{T})_{\bar{x}}$-equivariance structures, which are its descent data along the projection $\mathrm{L}(\mathrm{T})_{\bar{x}} \to (\mathrm{Hec}_\mathrm{T})_{\bar{x}}$.

As $\mathrm{L}_+(\mathrm{T})_{\bar{x}}$ acts trivially on $\bar{x}^\lambda$, the $\mathrm{L}_+(\mathrm{T})_{\bar{x}}$-equivariance structure on $e^*(i^\lambda)^*(\mathscr{G}) \cong \mathscr{G}_{\bar{x}^\lambda}$ is described by a rigidified A-torsor τ^λ over $\mathrm{L}_+(\mathrm{T})_{\bar{x}}$. It value at an S-point $t \in \mathrm{L}_+(\mathrm{T})_{\bar{x}}$ is the quotient of the lower circuit by the upper circuit of the square below:

$$\begin{array}{ccc} \mathscr{G}_{\varpi^\lambda \cdot t} & \longrightarrow & \mathscr{G}_{\varpi^\lambda} \otimes \mathscr{G}_t \\ \downarrow & & \downarrow \\ \mathscr{G}_{t \cdot \varpi^\lambda} & \longrightarrow & \mathscr{G}_t \otimes \mathscr{G}_{\varpi^\lambda} \end{array} \tag{3.11}$$

Here, the horizontal morphisms are the $\mathbb{E}_1$-monoidal structure of $\mathscr{G}$, the left vertical arrow is induced from $\varpi^\lambda \cdot t = t \cdot \varpi^\lambda$, and the right vertical arrow is the commutativity constraint of the 2-groupoid of A-gerbes.

It suffices to identify τ^λ with $\Psi^{b(-,\lambda)}$ as a rigidified A-torsor over $\mathrm{L}_+(\mathrm{T})_{\bar{x}}$, or equivalently as a rigidified A-torsor over $\mathrm{T}_{\bar{x}}$.

Recall that μ may be regarded as an $\mathbb{E}_1$-monoidal morphism $\mathrm{T} \to \mathrm{B}^3_{\mathrm{X}}\underline{\mathrm{A}}(1)$. The morphism $\mathrm{T} \times \mathrm{T} \to \mathrm{B}^2_{\mathrm{X}}\underline{\mathrm{A}}(1)$, sending a pair of S-points (t_1, t_2) to the quotient of the lower circuit by the upper circuit of the square below (with morphisms analogous to (3.11)):

$$\begin{array}{ccc} \mu_{t_1\cdot t_2} & \longrightarrow & \mu_{t_1} \otimes \mu_{t_2} \\ \downarrow & & \downarrow \\ \mu_{t_2\cdot t_1} & \longrightarrow & \mu_{t_2} \otimes \mu_{t_1} \end{array}$$

is rigidified along $e \times \mathrm{T}$ and $\mathrm{T} \times e$: it is a "bi-rigidified morphism" in the sense of [Zha22, §4.4.2]. As such, it agrees with the commutator of the corresponding extension of T by $\mathrm{B}^2_{\mathrm{X}}\underline{\mathrm{A}}(1)$, and equals $(\Psi \cup \Psi)^b$ by [Zha22, Corollary 4.7.6].

Finally, we observe that (3.10) is obtained from μ by taking sections over $\mathring{\mathrm{D}}_{\bar{x}}$, which is identified with the punctured formal disc of $\mathbb{G}_m$ at e. The fiber of $(\Psi \cup \Psi)^b$ at $\mathrm{T}_{\bar{x}} \times \varpi^\lambda$ then induces $\Psi^{b(-,\lambda)}$ along the canonical morphism $\Gamma(\mathring{\mathrm{D}}_{\bar{x}}, \underline{\mathrm{A}}(1)[2]) \to \underline{\mathrm{A}}[1]$. □

3.1.9. Before proving Proposition 3.1.2, we address the interaction between (3.7) and the group structure on Λ.

Let us start with the isomorphism (3.8).

For any section $\lambda \in \Lambda$, we have associated a section $\mu_*\mathscr{O}(\lambda\Delta)$ of $\Delta^!\underline{\mathrm{A}}(1)[4]$. (It reduces to the section denoted by $\mu_*\mathscr{O}(\Delta)$ for $\mathrm{T} = \mathbb{G}_m$ and $\lambda = 1$.) Recall that $\mu : \mathrm{B}_{\mathrm{X}}(\mathrm{T}) \to \mathrm{B}^4_{\mathrm{X}}\underline{\mathrm{A}}(1)$ is automatically "quadratic" in the sense of [Zha22, §4.7] and its associated symmetric form is given by the bi-rigidified morphism corresponding to b:

$$(\Psi \cup \Psi)^b : \mathrm{B}_{\mathrm{X}}(\mathrm{T}) \times \mathrm{B}_{\mathrm{X}}(\mathrm{T}) \to \mathrm{B}^4_{\mathrm{X}}\underline{\mathrm{A}}(1).$$

In particular, we find an isomorphism of sections of $\Delta^!\underline{\mathrm{A}}(1)[4]$:

$$\mu_*\mathscr{O}((\lambda_1 + \lambda_2)\Delta) \cong \mu_*\mathscr{O}(\lambda_1\Delta) \otimes \mu_*\mathscr{O}(\lambda_2\Delta) \otimes (\Psi \cup \Psi)^b_*(\mathscr{O}(\lambda_1\Delta), \mathscr{O}(\lambda_2\Delta)), \tag{3.12}$$

Under the purity isomorphism $\Delta^!\underline{\mathrm{A}}(1)[4] \cong \underline{\mathrm{A}}[2]$, (3.12) becomes:

$$e^*(i^{\lambda_1+\lambda_2})^*\mathscr{G}^{\{1\}} \cong e^*(i^{\lambda_1})^*\mathscr{G}^{\{1\}} \otimes e^*(i^{\lambda_2})^*\mathscr{G}^{\{1\}} \otimes \omega_{\mathrm{X}}^{b(\lambda_1,\lambda_2)}. \tag{3.13}$$

The isomorphism (3.8) intertwines (3.13) with the natural isomorphism:

$$\mathrm{F}_\vartheta(\lambda_1 + \lambda_2) \cong \mathrm{F}_\vartheta(\lambda_1) \otimes \mathrm{F}_\vartheta(\lambda_2) \otimes \omega_{\mathrm{X}}^{b(\lambda_1,\lambda_2)},$$

defined by the $\mathbb{E}_1$-monoidal structure on F and the natural identification of $\omega_{\mathrm{X}}^{\mathrm{Q}(\lambda_1+\lambda_2)}$ with the product $\omega_{\mathrm{X}}^{\mathrm{Q}(\lambda_1)} \otimes \omega_{\mathrm{X}}^{\mathrm{Q}(\lambda_2)} \otimes \omega_{\mathrm{X}}^{b(\lambda_1,\lambda_2)}$.

From this statement and the discreteness of the groupoid of rigidified A-gerbes over $\mathrm{B}_{\mathrm{X}^\lambda}(\mathrm{L}^{\{1\}}_+\mathrm{T})$, we deduce the analogous statement for (3.7).

Remark 3.1.10. In [Rei12] and [GL18], the isomorphism (3.13) is obtained from the factorization structure of $\mathscr{G}^{\{1,2\}}$, so the argument above could be seen as an alternative derivation of the same relationship.

Proof of Proposition 3.1.2. Denote by $\nu : \mathrm{Hec}^{\mathrm{I}}_{\mathrm{T}^\sharp} \to \mathrm{Hec}^{\mathrm{I}}_{\mathrm{T}}$ the map induced from $\mathrm{T}^\sharp \to \mathrm{T}$.

Let us embed the Satake categories into their analogues over the pairwise disjoint locus (corresponding to X^p for $p = \mathrm{id}_{\mathrm{I}}$) using (2.19):

$$\begin{array}{ccc} \mathrm{Sat}^{\mathrm{I}}_{\mathrm{T}^\sharp,\mathscr{G}^{\mathrm{I}}} & \subset & \mathrm{Sat}^{p}_{\mathrm{T}^\sharp,\mathscr{G}^{\mathrm{I}}} \\ \downarrow{\scriptstyle \nu_!} & & \downarrow{\scriptstyle \nu_!} \\ \mathrm{Sat}^{\mathrm{I}}_{\mathrm{T},\mathscr{G}^{\mathrm{I}}} & \subset & \mathrm{Sat}^{p}_{\mathrm{T},\mathscr{G}^{\mathrm{I}}} \end{array} \tag{3.14}$$

Claim: the right vertical functor in (3.14) is an equivalence.

Indeed, ν is an inclusion of connected components over X^p, so it suffices to show that any $\mathscr{F} \in \mathrm{Sat}^p_{T,\mathscr{G}^I}$ is supported on the union of $X^{\lambda^I} \times_{X^I} X^p$ with λ^I is a tuple of elements in $\Lambda^\sharp$.

The base change of $L^I_+(T)$ to a geometric point $\bar{x}^I \in X^p$ is identified with $\prod_{i\in I} L^{\{i\}}_+(T)_{\bar{x}^i}$. Given a tuple $\lambda^I = (\lambda^i)_{i\in I}$ with $\lambda^i \in \Lambda$, the A-gerbe $\mathscr{G}^I$ restricts to $\boxtimes_{i\in I}\mathscr{G}^{\{i\}}_{\bar{x}^i}$ over its classifying stack, viewed as a substack of $(\mathrm{Hec}^I_T)_{\bar{x}}$.

By Lemma 1.3.5 and Lemma 3.1.7, it suffices to prove:

$$H^0(\prod_{i\in I} T_{\bar{x}}, \boxtimes_{i\in I} \Psi^{b(-,\lambda_i)}) = 0 \quad \text{if some } \lambda_i \notin \Lambda^\sharp.$$

This follows from the vanishing of global sections of Ψ^a for $a \neq 0 \in A(-1)$, see §1.3.6.

To show that the left vertical functor in (3.14) is an equivalence, it remains to show that it is essentially surjective. Since the essential image of (2.19) is closed under direct summands, an object $\mathscr{F} \in \mathrm{Sat}^p_{T,\mathscr{G}^I}$ belongs to $\mathrm{Sat}^I_{T,\mathscr{G}^I}$ if and only if its restriction to each $X^{\lambda^I} \times_{X^I} X^p$ extends to a lisse E-sheaf on X^{λ^I}. The same assertion for $T^\sharp$ then concludes the proof. □

Remark 3.1.11. The proof of Proposition 3.1.2 also shows that $\mathrm{Sat}^I_{T^\sharp}$ is equivalent to the abelian category $\mathrm{Perv}(\mathrm{Gr}^I_{T^\sharp})_{/X^I}$ of perverse, universally locally acyclic E-sheaves on $\mathrm{Gr}^I_{T^\sharp}$, i.e. the $L^I_+(T^\sharp)$-equivariance condition is automatic.

The analogous statement for T instead of $T^\sharp$ is emphatically false.

3.2. Split tori: equivalence.

3.2.1. We continue to suppose that $G = T$ is a split torus with sheaf of cocharacters Λ. Furthemore, we assume that the symmetric form b vanishes, i.e. $\Lambda^\sharp = \Lambda$.

Under this assumption, the étale metaplectic cover $\mu : B_X(T) \to B^4_X\underline{A}(1)$ canonically lifts to an $\mathbb{E}_\infty$-monoidal morphism, see [Zha22, §4.6].

The metaplectic dual data (H, F) admit a concrete description:

1. H is the dual torus $\check{T}$ over E;
2. $F : \Lambda \to B^2_X(\underline{A})$ is (3.6), with $\mathbb{E}_\infty$-monoidal structure induced from μ_* (3.5).

Our current goal is to prove the geometric Satake equivalence for such (T, μ).

Proposition 3.2.2. *For a nonempty finite set* I*, there is a canonical equivalence of tensor categories:*

$$\mathrm{Sat}^I_{T,\mathscr{G}^I} \cong \mathrm{Rep}^I_{\check{T}^I, F^I_\vartheta}. \tag{3.15}$$

3.2.3. For a nonempty finite set I, we regard Gr^I_T as an étale sheaf of abelian groups over X^I, being defined by the quotient $L^I(T)/L^I_+(T)$. The collection of closed immersions (3.2) is gathered into a morphism of sheaves of abelian groups $\Lambda^I \to \mathrm{Gr}^I_T$, the source being the external direct sum of Λ over $i \in I$.

Since μ admits an $\mathbb{E}_\infty$-monoidal structure, the commutative diagram (for $G = T$) lifts to a commutative diagram of étale sheaves of $\mathbb{E}_\infty$-monoids over X^I. In particular, $\mathscr{G}^I$ may be viewed an $\mathbb{E}_\infty$-monoidal morphism $\mathrm{Gr}^I_T \to B^2_{X^I}(\underline{A})$.

Composing these two morphisms, we obtain an $\mathbb{E}_\infty$-monoidal morphism:

$$\Lambda^I \to B^2_{X^I}(\underline{A}), \quad \lambda^I \mapsto e^*(i^{\lambda^I})^*\mathscr{G}^I. \tag{3.16}$$

Lemma 3.2.4. *The* $\mathbb{E}_\infty$*-monoidal morphism* (3.16) *is canonically identified with:*

$$F^I_\vartheta : \Lambda^I \to B^2_{X^I}(\underline{A}), \quad (\lambda^i)_{i\in I} \mapsto \boxtimes_{i\in I} F_\vartheta(\lambda^i).$$

Proof. For each $i \in \mathrm{I}$, let $\gamma^i : \Gamma_{x^i} \cong \mathrm{X}^{\mathrm{I}} \to \mathrm{X}^{\mathrm{I}} \times \mathrm{X}$ denote the graph of the ith projection $x^i : \mathrm{X}^{\mathrm{I}} \to \mathrm{X}$. The union of the graphs is denoted by $\gamma : \Gamma_{x^{\mathrm{I}}} \to \mathrm{X}^{\mathrm{I}} \times \mathrm{X}$. (We are following the notations of §2.1, viewing $x^{\mathrm{I}} = (x^i)_{i\in\mathrm{I}}$ as an X^{I}-point of X.) We also have the projections $p_1 : \mathrm{X}^{\mathrm{I}} \times \mathrm{X} \to \mathrm{X}^{\mathrm{I}}$, $p_2 : \mathrm{X}^{\mathrm{I}} \times \mathrm{X} \to \mathrm{X}$ onto the X^{I}, respectively the last factor.

Recall that sheaves of $\mathbb{E}_\infty$-monoids are equivalent to sheaves of connective spectra, where the functor B corresponds to suspension [1].

The pullback $p_2^*(\mu)$ defines a morphism of sheaves of connective spectra $p_2^*(\mu) : \mathrm{T}[1] \to \underline{\mathrm{A}}(1)[4]$ over X^{I}, and induces a commutative diagram of such:

$$\begin{array}{ccccc} \Lambda^{\mathrm{I}} & \longrightarrow & \bigoplus_{i\in\mathrm{I}}(\gamma^i)^!\mathrm{T}[1] & \xrightarrow{p_2^*(\mu)} & \bigoplus_{i\in\mathrm{I}}(\gamma^i)^!\underline{\mathrm{A}}(1)[4] \\ & & \downarrow\Sigma & & \downarrow\Sigma \\ & & (p_1\cdot\gamma)_*\gamma^!\mathrm{T}[1] & \xrightarrow{p_2^*(\mu)} & (p_1\cdot\gamma)_*\gamma^!\underline{\mathrm{A}}(1)[4] \\ & & & & \downarrow \\ & & & & \underline{\mathrm{A}}[2] \end{array} \tag{3.17}$$

Here, the first horizontal morphism sends $(\lambda^i)_{i\in\mathrm{I}}$ to the section $(\mathscr{O}(\lambda^i\Gamma_{x^i}))_{i\in\mathrm{I}}$, and the last vertical morphism is the one from (2.10).

The upper circuit of (3.17) is the morphism (3.16). The lower circuit of (3.17) is the external sum of the morphisms $\Lambda \to \underline{\mathrm{A}}[2]$ defined by (3.16) for the singleton $\{i\}$. The construction thus reduces to the case $\mathrm{I} = \{1\}$, where it is given by (3.8).

To see that (3.8) is compatible with the $\mathbb{E}_\infty$-monoidal structures, we first appeal to its compatibility with the $\mathbb{E}_1$-monoidal structures addressed in §3.1.9. Its compatibility with commutativity constraints is then a *condition* and can be checked étale locally on X. This follows from the case $\mathrm{X} = \mathbb{A}^1$, where the agreement is definitional (see §3.1.5). □

Proof of Proposition 3.2.2. Let us construct the tensor equivalence (3.15).

Consider the product map $m : \prod_{i\in\mathrm{I}} \mathrm{Gr}_{\mathrm{T}}^{\{i\}} \to \mathrm{Gr}_{\mathrm{T}}^{\mathrm{I}}$ of ind-schemes over X^{I}. (Any ordering $\mathrm{I} \cong \{1, \cdots, k\}$ realizes $\prod_{i\in\mathrm{I}} \mathrm{Gr}_{\mathrm{T}}^{\{i\}}$ as the ind-scheme $\widetilde{\mathrm{Gr}}_{\mathrm{T}}^{\{1\},\cdots,\{k\}}$ of §2.1.5 and m the composition of all modifications.)

Note that the reduced sub-indscheme of $\prod_{i\in\mathrm{I}} \mathrm{Gr}_{\mathrm{T}}^{\{i\}}$ represents the étale sheaf Λ^{I}. Using Lemma 3.2.4, we find functors of tensor categories:

$$\begin{aligned} \mathrm{Rep}^{\mathrm{I}}_{\check{\mathrm{T}},\mathrm{F}^{\mathrm{I}}_{\vartheta}} &\cong \bigoplus_{\lambda^{\mathrm{I}}\in\Lambda^{\mathrm{I}}} \mathrm{Lis}_{\mathrm{F}^{\mathrm{I}}_{\vartheta}(\lambda^{\mathrm{I}})}(\mathrm{X}^{\mathrm{I}}) \\ &\cong \mathrm{Perv}_{\mathscr{G}^{\mathrm{I}}}(\prod_{i\in\mathrm{I}} \mathrm{Gr}_{\mathrm{T}}^{\{i\}})_{/\mathrm{X}^{\mathrm{I}}} \xrightarrow{m_!} \mathrm{Perv}_{\mathscr{G}^{\mathrm{I}}}(\mathrm{Gr}_{\mathrm{T}}^{\mathrm{I}})_{/\mathrm{X}^{\mathrm{I}}} \cong \mathrm{Sat}^{\mathrm{I}}_{\mathrm{T},\mathscr{G}^{\mathrm{I}}}, \end{aligned} \tag{3.18}$$

where $\mathrm{Sat}^{\mathrm{I}}_{\mathrm{T},\mathscr{G}^{\mathrm{I}}}$ is equipped with the tensor structure induced from the $\mathbb{E}_\infty$-monoidal morphism $\mathscr{G}^{\mathrm{I}} : \mathrm{Gr}_{\mathrm{T}}^{\mathrm{I}} \to \mathrm{B}^2_{\mathrm{X}^{\mathrm{I}}}(\underline{\mathrm{A}})$. This tensor structure naturally extends the *convolution* monoidal structure $\circ_{\mathrm{X}}$ on $\mathrm{Sat}^{\mathrm{I}}_{\mathrm{T},\mathscr{G}^{\mathrm{I}}}$.

Equiped with this tensor structure, $\mathrm{Sat}^{\mathrm{I}}_{\mathrm{T},\mathscr{G}^{\mathrm{I}}}$ still coincides with $(\mathrm{Sat}^{\mathrm{I}}_{\mathrm{T},\mathscr{G}^{\mathrm{I}}}, \star_{\mathrm{X}}, e_!(\underline{\mathrm{E}}))$ defined by the fusion product, as it lifts the latter to an $\mathbb{E}_\infty$*-monoid* in the 2-category of symmetric monoidal categories, *c.f.* Remark 2.3.13.

It remains to show that the functor $m_!$ in (3.18) is an equivalence. By universal local acyclicity, both categories embed fully faithfully in their analogues over the pairwise disjoint locus $\mathrm{X}^p \subset \mathrm{X}^{\mathrm{I}}$ (see §2.3.10), where they are both equivalent to $\mathrm{Sat}^p_{\mathrm{T},\mathscr{G}^{\mathrm{I}}}$. This implies that $m_!$ is fully faithful. To see that it is essentially surjective, we observe that an object $\mathscr{F} \in \mathrm{Sat}^p_{\mathrm{T},\mathscr{G}^{\mathrm{I}}}$

belongs to $\mathrm{Sat}^{\mathrm{I}}_{\mathrm{T},\mathscr{G}^{\mathrm{I}}}$ if and only if its restriction to each $\mathrm{X}^{\lambda^{\mathrm{I}}} \times_{\mathrm{X}^{\mathrm{I}}} \mathrm{X}^p$ extends to a lisse E-sheaf on $\mathrm{X}^{\lambda^{\mathrm{I}}}$, as the essential image of (2.19) is closed under direct summands. □

3.2.5. We relax the condition $b = 0$, i.e. μ stands for any étale metaplectic cover of T.

We shall construct the geometric Satake equivalence (2.23) for split tori.

Construction of (2.23) *for* G = T. Let $\mathrm{T}^\sharp \to \mathrm{T}$ be the isogeny of split tori corresponding to $\Lambda^\sharp \subset \Lambda$ as in §3.1.1.

The metaplectic dual pair (H, F) is precisely the pair $\mathrm{H} = \check{\mathrm{T}}^\sharp$, $\mathrm{F} : \Lambda^\sharp \to \mathrm{B}^2_{\mathrm{X}}(\underline{\mathrm{A}})$ associated to $\mathrm{T}^\sharp$ and the restriction of μ as in §3.2.1.

The equivalence (2.23) is thus the composition of the inverse of (3.1) with (3.15):

$$\mathrm{Sat}^{\mathrm{I}}_{\mathrm{T},\mathscr{G}^{\mathrm{I}}} \cong \mathrm{Sat}^{\mathrm{I}}_{\mathrm{T}^\sharp,\mathscr{G}^{\mathrm{I}}} \cong \mathrm{Rep}^{\mathrm{I}}_{\mathrm{H}^{\mathrm{I}},\mathrm{F}^{\mathrm{I}}_{\vartheta}}.$$

Theorem 2.4.4 for tori is proved. □

3.3. The abelian category $\mathrm{Sat}_{\mathrm{G},\mathscr{G}}$.

3.3.1. Let us now turn to the context where $\mathrm{G} \to \mathrm{X}$ is split reductive with chosen Borel subgroup $\mathrm{B} \subset \mathrm{G}$. Choose furthermore a splitting of $\mathrm{B} \twoheadrightarrow \mathrm{T}$ and view T as a maximal torus of G.

The goal of this subsection is to determine $\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}$ as an abelian category for $\mathrm{I} = \{1\}$. To lighten the notations, we omit the superscript I in this subsection.

3.3.2. Write $\Lambda^+ \subset \Lambda$ for the submonoid consisting of dominant cocharacters. Each $\lambda \in \Lambda^+$ determines a Schubert cell $\mathrm{Gr}^\lambda_{\mathrm{G}}$ as the $\mathrm{L}_+\mathrm{G}$-orbit of X^λ, embedded in $\mathrm{Gr}^\lambda_{\mathrm{G}}$ along (3.2) and the closed immersion $\mathrm{Gr}_{\mathrm{T}} \subset \mathrm{Gr}_{\mathrm{G}}$.

The closure of $\mathrm{Gr}^\lambda_{\mathrm{G}}$ is identified with $\mathrm{Gr}^{\le\lambda}_{\mathrm{G}} := \bigcup_{\lambda_1\le\lambda} \mathrm{Gr}^{\lambda_1}_{\mathrm{G}}$. Denote by $j^\lambda : \mathrm{Gr}^\lambda_{\mathrm{G}} \subset \mathrm{Gr}^{\le\lambda}_{\mathrm{G}}$ the open immersion.

Write $\mathrm{P}^\lambda \subset \mathrm{G}$ for the standard parabolic subgroup corresponding to the simple roots annihilated by λ. It has Levi quotient M^λ. The quotient map $\mathrm{L}_+\mathrm{G} \to \mathrm{G}$ induces a map on their homogeneous spaces $\mathrm{Gr}^\lambda_{\mathrm{G}} \to \mathrm{G}/\mathrm{P}^\lambda \to \mathrm{G}/\mathrm{M}^\lambda$, see [Zhu17, §2.1]. Its quotient by $\mathrm{L}_+\mathrm{G}$ on the source and G on the target defines a map p_{M^λ}, fitting into the following diagram:

$$\begin{array}{ccccc} & & \mathrm{X}^\lambda & \cong & \mathrm{X}^\lambda \\ & & \downarrow e & & \downarrow \cong \\ \mathrm{Hec}_{\mathrm{G}} & \xleftarrow{i^\lambda} & \mathrm{L}_+\mathrm{G}\backslash\mathrm{Gr}^\lambda_{\mathrm{G}} & \xrightarrow{p} & \mathrm{X} \\ & & \downarrow p_{\mathrm{M}^\lambda} & & \downarrow \cong \\ & & \mathrm{B}_{\mathrm{X}}(\mathrm{M}^\lambda) & \longrightarrow & \mathrm{X} \end{array} \tag{3.19}$$

3.3.3. Since M^λ is reductive, the groupoid of rigidified A-gerbes over $\mathrm{B}_{\mathrm{X}}(\mathrm{M}^\lambda)$ is identified with the discrete abelian group $\mathrm{Hom}(\pi_1(\mathrm{M}^\lambda), \mathrm{A}(-1))$. This follows, for example, from [Zha22, Proposition 5.1.11].

The natural map $\mathrm{T} \to \mathrm{M}^\lambda$ induces a surjection $\Lambda \to \pi_1(\mathrm{M}^\lambda)$ whose kernel is spanned by simple coroots whose associated roots are annihilated by λ. Via this map, rigidified A-gerbes over $\mathrm{B}_{\mathrm{X}}(\mathrm{M}^\lambda)$ form an abelian subgroup of those over $\mathrm{B}_{\mathrm{X}}(\mathrm{T})$.

Recall the rigidified A-gerbe $\Psi^{b(-,\lambda)}$ defined in §3.1.4. The identity (1.22) implies that $\Psi^{b(-,\lambda)}$ defines a rigidified A-gerbe over $\mathrm{B}_{\mathrm{X}}(\mathrm{M}^\lambda)$.

3.3.4. The following Lemma is a generalization (and corollary) of Lemma 3.1.7. It is an analogue of [FL10, Lemma 2.4] in étale cohomology.

Lemma 3.3.5. *For each* $\lambda \in \Lambda^+$*, there is a canonical isomorphism of* A*-gerbes, in reference to the morphisms in* (3.19)*:*

$$(i^\lambda)^*\mathscr{G} \cong p^*\mathrm{F}_\vartheta(\lambda) \otimes (p_{\mathrm{M}^\lambda})^*(\Psi^{b(-,\lambda)}). \tag{3.20}$$

(For $\lambda = 0$*, this is the identity automorphism of the trivial* A*-gerbe.)*

Proof. Since the kernel of $\mathrm{L}_+\mathrm{G} \to \mathrm{G}$ is pro-unipotent, pulling back by p_{M^λ} defines an equivalence on the groupoid of (rigidified) A-gerbes.

By Lemma 3.1.7, we already have an isomorphism:

$$e^*(i^\lambda)^*\mathscr{G} \cong \mathrm{F}_\vartheta(\lambda).$$

It remains to show that $(i^\lambda)^*\mathscr{G} \otimes p^*\mathrm{F}_\vartheta(\lambda)^{\otimes -1}$ equals $\Psi^{b(-,\lambda)}$ as rigidified A-gerbes over $\mathrm{B_X}(\mathrm{M}^\lambda)$. This statement can be proved after pulling back along $\mathrm{B_X}(\mathrm{T}) \to \mathrm{B_X}(\mathrm{M}^\lambda)$, where it again reduces to Lemma 3.1.7. □

3.3.6. Let $\lambda \in \Lambda^{\sharp,+} := \Lambda^\sharp \cap \Lambda^+$. The isomorphism (3.20) shows that pulling back along the projection p carries the A-gerbe $\mathrm{F}_\vartheta(\lambda)$ to the restriction of $\mathscr{G}$.

In particular, any $\mathrm{F}_\vartheta(\lambda)$-twisted E-local system $\mathscr{E}$ over X pulls back to a $\mathscr{G}$-twisted $\mathrm{L}_+\mathrm{G}$-equivariant E-local system $p^*(\mathscr{E})$ over $\mathrm{Gr}_\mathrm{G}^\lambda$. Up to cohomological shift and Tate twist, we may form its intermediate extension along j^λ as an object of $\mathrm{Sat}_{\mathrm{G},\mathscr{G}}$:

$$\mathrm{IC}_\mathscr{E} := (j^\lambda)_{!*}p^*\mathscr{E}(\langle\check{\rho},\lambda\rangle)[\langle 2\check{\rho},\lambda\rangle] \quad \in \mathrm{Sat}_{\mathrm{G},\mathscr{G}}.$$

Here, $\langle\check{\rho},\lambda\rangle \in \frac{1}{2}\mathbb{Z}$ and the Tate twist is formed with the aid of $\underline{\mathrm{E}}(\frac{1}{2})$.

Proposition 3.3.7. *The functor below is an equivalence of* E*-linear abelian categories:*

$$\bigoplus_{\lambda\in\Lambda^{\sharp,+}} \mathrm{Lis}_{\mathrm{F}_\vartheta(\lambda)}(\mathrm{X}) \to \mathrm{Sat}_{\mathrm{G},\mathscr{G}}, \quad (\mathscr{E}^\lambda) \mapsto \bigoplus_{\lambda\in\Lambda^{\sharp,+}} \mathrm{IC}_{\mathscr{E}^\lambda}. \tag{3.21}$$

Proof. For each $\lambda \in \Lambda^{\sharp,+}$, the morphism p in (3.19) is smooth with connected fibers. It follows that (3.21) restricts to a fully faithful functor on each summand.

By the definition of intermediate extensions, the images of distinct summands under (3.21) are orthogonal. Thus (3.21) is fully faithful. It remains to show that it is also essentially surjective, i.e. any $\mathscr{F} \in \mathrm{Sat}_{\mathrm{G},\mathscr{G}}$ is a direct sum of objects of the form $\mathrm{IC}_{\mathscr{E}^\lambda}$ over $\lambda \in \Lambda^{\sharp,+}$.

For each $\lambda \in \Lambda^+$, we let $\mathscr{F}^\lambda \in \mathrm{Lis}_{\mathrm{F}_\vartheta(\lambda)}(\mathrm{X},\mathrm{E})$ denote the restriction of $\mathscr{F}$ to X^λ. If $\lambda \notin \Lambda^{\sharp,+}$, it follows from Lemma 1.3.5 and Lemma 3.3.5 that $\mathscr{F}^\lambda = 0$, so the restriction of $\mathscr{F}$ to $\mathrm{Gr}_\mathrm{G}^\lambda$ vanishes by $\mathrm{L}_+\mathrm{G}$-equivariance. If $\lambda \in \Lambda^{\sharp,+}$, the restriction of $\mathscr{F}$ to $\mathrm{Gr}_\mathrm{G}^\lambda$ is given by $p^*(\mathscr{F}^\lambda)$. Thus $\mathscr{F}$ is an iterated extension of the objects $\mathrm{IC}_{\mathscr{F}^\lambda}$ over $\lambda \in \Lambda^{\sharp,+}$.

It remains to show that for $\lambda_1 \neq \lambda_2$, the images of $\mathrm{Lis}_{\mathrm{F}_\vartheta(\lambda_1)}(\mathrm{X})$ and $\mathrm{Lis}_{\mathrm{F}_\vartheta(\lambda_1)}(\mathrm{X})$ in $\mathrm{Sat}_{\mathrm{G},\mathscr{G}}$ have no nonsplit extensions. This is proved in the same way as the non-metaplectic setting, by reducing it to Lusztig's parity vanishing of the stalks of the intersection cohomology complex, see [Zhu17, Proposition 5.1.1] and the references therein. □

Remark 3.3.8. For G = T, the decomposition (3.21) coincides with (3.18) appearing in the proof of Proposition 3.2.2.

Contrary to the case of tori, (3.21) is *incompatible* with the monoidal structure on $\mathrm{Sat}_{\mathrm{G},\mathscr{G}}$, i.e. the monoidal product of two homogeneous objects is in general inhomogeneous.

3.4. Constant terms: construction.

3.4.1. Suppose that $\mathrm{P} \subset \mathrm{G}$ is a standard parabolic subgroup with unipotent radical $\mathrm{N_P} \subset \mathrm{P}$ and Levi quotient $\mathrm{P} \twoheadrightarrow \mathrm{M}$.

The restriction of μ to B(P) canonically descends to an étale metaplectic cover of M.

3.4.2. For a nonempty finite set I, the construction of §2.2 for M produces an A-gerbe $\mathscr{G}^{\mathrm{I}}_{\mathrm{M}}$ over $\mathrm{Hec}^{\mathrm{I}}_{\mathrm{M}}$. We have an isomorphism of A-gerbes:

$$p^*(\mathscr{G}^{\mathrm{I}}) \cong q^*(\mathscr{G}^{\mathrm{I}}_{\mathrm{M}}) \tag{3.22}$$

along the canonical morphisms $p : \mathrm{Hec}^{\mathrm{I}}_{\mathrm{P}} \to \mathrm{Hec}^{\mathrm{I}}_{\mathrm{G}}$ and $q : \mathrm{Hec}^{\mathrm{I}}_{\mathrm{P}} \to \mathrm{Hec}^{\mathrm{I}}_{\mathrm{M}}$.

Define the *(naïve) constant term functor* to be the following functor of E-linear stable ∞-categories, using the isomorphism (3.22):

$$\mathrm{CT}^{\mathrm{I}}_{\mathrm{P}} : \mathrm{Shv}_{\mathscr{G}^{\mathrm{I}}}(\mathrm{Hec}^{\mathrm{I}}_{\mathrm{G}}) \to \mathrm{Shv}_{\mathscr{G}^{\mathrm{I}}_{\mathrm{M}}}(\mathrm{Hec}^{\mathrm{I}}_{\mathrm{M}}), \quad \mathscr{F} \mapsto q_! p^*(\mathscr{F}). \tag{3.23}$$

3.4.3. Since the connected components of $\mathrm{Hec}^{\mathrm{I}}_{\mathrm{M}}$ are enumerated by $\pi_1(\mathrm{M})$, the sum of positive roots occurring in N_{P} defines a character $2\check{\rho}_{\mathrm{P}} : \pi_1(\mathrm{M}) \to \mathbb{Z}$, which we view as a locally constant function on $\mathrm{Hec}^{\mathrm{I}}_{\mathrm{M}}$. (In particular, $2\check{\rho}_{\mathrm{B}} = 2\check{\rho}$.)

We shall adjust (3.23) by a cohomological shift by $2\check{\rho}_{\mathrm{P}}$ and Tate twist by $\check{\rho}_{\mathrm{P}}$ (with the aid of $\underline{\mathrm{E}}(\frac{1}{2})$). The result will be a tensor functor on the Satake categories with modified commutativity constraints, as defined in §2.4.

Proposition 3.4.4. *The functor* (3.23) *induces an exact tensor functor:*

$$\mathrm{CT}^{\mathrm{I}}_{\mathrm{P}}(\check{\rho}_{\mathrm{P}})[2\check{\rho}_{\mathrm{P}}] : {}^{+}\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}} \to {}^{+}\mathrm{Sat}^{\mathrm{I}}_{\mathrm{M},\mathscr{G}^{\mathrm{I}}_{\mathrm{M}}}. \tag{3.24}$$

Proof. The proof is identical to its non-metaplectic counterpart and follows from Braden's hyperbolic localization theorem. We briefly indicate the argument.

Claim: $\mathrm{CT}^{\mathrm{I}}_{\mathrm{P}}(\check{\rho}_{\mathrm{P}})[2\check{\rho}_{\mathrm{P}}]$ carries the abelian subcategory $\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}$ to $\mathrm{Sat}^{\mathrm{I}}_{\mathrm{M},\mathscr{G}^{\mathrm{I}}_{\mathrm{M}}}$.

Indeed, after establishing the claim, the tensor structure on $\mathrm{CT}^{\mathrm{I}}_{\mathrm{P}}(\check{\rho}_{\mathrm{P}})[2\check{\rho}_{\mathrm{P}}]$ arises from its commutation with the external fusion products (see §2.3.10). Note that due to the degree shift $[2\check{\rho}_{\mathrm{P}}]$, the Koszul sign rule implies that $\mathrm{CT}^{\mathrm{I}}_{\mathrm{P}}(\check{\rho}_{\mathrm{P}})[2\check{\rho}_{\mathrm{P}}]$ is compatible with exchanging factors *after* modifying the commutativity constraints.

To prove the claim, we may split the Levi quotients of B and P and view M as a subgroup of G contaning the maximal torus T. Let $\mathrm{P}^- \subset \mathrm{G}$ be the parabolic subgroup opposite to P. The local Hecke stack $\mathrm{Hec}^{\mathrm{I}}_{\mathrm{P}^-}$ is equipped with projections p^-, q^- to $\mathrm{Hec}^{\mathrm{I}}_{\mathrm{G}}$, respectively $\mathrm{Hec}^{\mathrm{I}}_{\mathrm{M}}$. There is a functor:

$$\mathrm{CT}^{\mathrm{I},!}_{\mathrm{P}^-} : \mathrm{Shv}_{\mathscr{G}^{\mathrm{I}}}(\mathrm{Hec}^{\mathrm{I}}_{\mathrm{G}}) \to \mathrm{Shv}_{\mathscr{G}^{\mathrm{I}}_{\mathrm{M}}}(\mathrm{Hec}^{\mathrm{I}}_{\mathrm{M}}), \quad \mathscr{F} \mapsto (q^-)_*(p^-)^!(\mathscr{F}). \tag{3.25}$$

Braden's theorem (as stated in [DG14]) identifies the functor (3.23) with (3.25). Its formation commutes with base change along $\mathrm{S} \to \mathrm{X}^{\mathrm{I}}$ for any k-scheme S.

To show that $\mathrm{CT}^{\mathrm{I}}_{\mathrm{P}}$ preserves universal local acyclicity relative to X^{I}, we base change to the spectrum of a rank-1 valuation ring and observe that the characterization [HS21, Theorem 4.4(iv)] holds for $\mathrm{CT}^{\mathrm{I},!}_{\mathrm{P}^-}$.

To show that $\mathrm{CT}^{\mathrm{I}}_{\mathrm{P}}(\check{\rho}_{\mathrm{P}})[2\check{\rho}_{\mathrm{P}}]$ is t-exact with respect to the perverse t-structure relative to X^{I}, we base change to any geometric point of X^{I} and reduce to the case $\mathrm{I} = \{1\}$. The argument of [MV07, §3] then shows that $\mathrm{CT}^{\{1\}}_{\mathrm{P}}(\check{\rho}_{\mathrm{P}})[2\check{\rho}_{\mathrm{P}}]$ is right t-exact and $\mathrm{CT}^{\{1\},!}_{\mathrm{P}^-}(\check{\rho}_{\mathrm{P}})[2\check{\rho}_{\mathrm{P}}]$ is left t-exact. ☐

Remark 3.4.5. The tensor functor (3.24) is compatible with compositions. More precisely, for a parabolic subgroup $\mathrm{P}_1 \subset \mathrm{M}$ with Levi quotient $\mathrm{P}_1 \twoheadrightarrow \mathrm{M}_1$, the composition of $\mathrm{CT}^{\mathrm{I}}_{\mathrm{P}_1}(\check{\rho}_{\mathrm{P}_1})[2\check{\rho}_{\mathrm{P}_1}]$ with $\mathrm{CT}^{\mathrm{I}}_{\mathrm{P}}(\check{\rho}_{\mathrm{P}})[2\check{\rho}_{\mathrm{P}}]$ is canonically isomorphic to $\mathrm{CT}^{\mathrm{I}}_{\mathrm{P}_0}(\check{\rho}_{\mathrm{P}_0})[2\check{\rho}_{\mathrm{P}_0}]$, where P_0 denotes the parabolic subgroup $\mathrm{P}_0 \coloneqq \mathrm{P} \times_{\mathrm{M}} \mathrm{P}_1 \subset \mathrm{G}$.

Remark 3.4.6. The Levi quotient of any Borel subgroup $\mathrm{B} \subset \mathrm{G}$ is identified with the universal Cartan T, and the tensor functor $\mathrm{CT}^{\mathrm{I}}_{\mathrm{B}}(\check{\rho})[2\check{\rho}]$ (3.24) is independent of the choice of $\mathrm{B} \subset \mathrm{G}$. More precisely, given two Borel subgroups $\mathrm{B}_1, \mathrm{B}_2 \subset \mathrm{G}$, there is a canonical isomorphism of tensor functors:

$$\mathrm{CT}^{\mathrm{I}}_{\mathrm{B}_1}(\check{\rho})[2\check{\rho}] \cong \mathrm{CT}^{\mathrm{I}}_{\mathrm{B}_2}(\check{\rho})[2\check{\rho}], \tag{3.26}$$

subject to the natural compatibility for three Borel subgroups.

Let us first construct (3.26) subject to the choice of a section $g \in \mathrm{G}$ such that the inner automorphism int_g of G carries B_1 to B_2.

Indeed, the moduli description of $\mathrm{Hec}^{\mathrm{I}}_{\mathrm{G}}$ shows that int_g induces the identity automorphism on $\mathrm{Hec}^{\mathrm{I}}_{\mathrm{G}}$. Thus pulling back along int_g yields a commutative diagram:

$$\begin{array}{ccc} {}^{+}\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}} & \xrightarrow{\mathrm{id}} & {}^{+}\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}} \\ {\scriptstyle \mathrm{CT}^{\mathrm{I}}_{\mathrm{B}_2}(\check{\rho})[2\check{\rho}]}\downarrow & & \downarrow{\scriptstyle \mathrm{CT}^{\mathrm{I}}_{\mathrm{B}_1}(\check{\rho})[2\check{\rho}]} \\ \mathrm{Sat}^{\mathrm{I}}_{\mathrm{T}_1,\mathscr{G}^{\mathrm{I}}_{\mathrm{T}_1}} & \xrightarrow{(\mathrm{int}_g)^*} & \mathrm{Sat}^{\mathrm{I}}_{\mathrm{T}_2,\mathscr{G}^{\mathrm{I}}_{\mathrm{T}_2}} \end{array} \tag{3.27}$$

Here, T_1 (resp. T_2) denotes the maximal quotient torus of B_1 (resp. B_2), so the isomorphism $\mathrm{int}_g : \mathrm{T}_1 \cong \mathrm{T}_2$ is encoded in the definition of T. The commutative diagram (3.27) yields an isomorphism F_g of the two functors in (3.26).

It remains to prove that for $\mathrm{B} = \mathrm{B}_1 = \mathrm{B}_2$ and $g \in \mathrm{B}$, the isomorphism F_g equals the identity automorphism of $\mathrm{CT}^{\mathrm{I}}_{\mathrm{B}}(\check{\rho})[2\check{\rho}]$, where the endofunctor $(\mathrm{int}_g)^*$ of $\mathrm{Sat}^{\mathrm{I}}_{\mathrm{T},\mathscr{G}^{\mathrm{I}}_{\mathrm{T}}}$ is trivialized as int_g induces the identity on T.

For this statement, we construct the automorphisms F_g as a family over $g \in \mathrm{B}$, i.e. they define an automorphism of the functor $\underline{\mathrm{E}}_{\mathrm{B}} \boxtimes \mathrm{CT}^{\mathrm{I}}_{\mathrm{B}}(\check{\rho})[2\check{\rho}]$ valued in E-sheaves over $\mathrm{B} \times \mathrm{Hec}^{\mathrm{I}}_{\mathrm{T}}$. This automorphism equals the identity, because it restricts to the identity over $e \times \mathrm{Hec}^{\mathrm{I}}_{\mathrm{T}}$ and B is smooth and connected.

3.5. Constant terms: vanishing.

3.5.1. In this subsection, we specialize to the case $\mathrm{I} = \{1\}$ and study the behavior of the constant term functor (3.23) associated to the Borel subgroup $\mathrm{B} \subset \mathrm{G}$.

We denote the Weyl group of G by W. It canonically acts on Λ. As in §3.3, we temporarily drop the superscript $\{1\}$.

3.5.2. Note that for any $\lambda \in \Lambda^{\sharp}$ and $w \in \mathrm{W}$, the difference $\lambda - w(\lambda)$ belongs to $\Lambda^{\sharp,r}$.

Recall that the $\mathbb{E}_\infty$-monoidal morphism $\mathrm{F} : \Lambda^{\sharp} \to \mathrm{B}^2_{\mathrm{X}}(\underline{\mathrm{A}})$ is canonically trivialized over $\Lambda^{\sharp,r}$ ([Zha22, 6.1.5]). Since Q vanishes over $\Lambda^{\sharp,r}$, the $\mathbb{Z}$-linear morphism $\omega^{\mathrm{Q}}_{\mathrm{X}} : \Lambda^{\sharp} \to \mathrm{B}^2_{\mathrm{X}}(\underline{\mathrm{A}})$ of §1.6.15 is likewise trivialized over $\Lambda^{\sharp,r}$.

Therefore, F_{ϑ} is equipped with a canonical W-invariance structure.

3.5.3. According to Proposition 3.3.7, the functor $\mathrm{CT}_{\mathrm{B}}(\check{\rho})[2\check{\rho}]$ is the direct sum of functors indexed by pairs of elements $\lambda \in \Lambda^{\sharp,+}, \lambda_1 \in \Lambda^{\sharp}$:

$$\begin{aligned} \mathrm{CT}^{\lambda,\lambda_1}_{\mathrm{B}}(\check{\rho})[2\check{\rho}] : \mathrm{Lis}_{\mathrm{F}_{\vartheta}(\lambda)}(\mathrm{X}) \subset {}^{+}\mathrm{Sat}_{\mathrm{G},\mathscr{G}} & \\ \xrightarrow{\mathrm{CT}_{\mathrm{B}}(\check{\rho})[2\check{\rho}]} \mathrm{Sat}_{\mathrm{T},\mathscr{G}_{\mathrm{T}}} \twoheadrightarrow \mathrm{Lis}_{\mathrm{F}_{\vartheta}(\lambda_1)}(\mathrm{X}).& \end{aligned} \tag{3.28}$$

The next Proposition describes the behavior of (3.28). Its proof will occupy the remainder of this subsection.

Proposition 3.5.4. *Let $\lambda \in \Lambda^{\sharp,+}$ and $\lambda_1 \in \Lambda^{\sharp}$. The following statements hold:*

(1) if $\lambda_1 \in \mathrm{W}\lambda$, then $\mathrm{CT}^{\lambda,\lambda_1}_{\mathrm{B}}(\check{\rho})[2\check{\rho}]$ is equivalent to the identity functor (in reference to the canonical W-invariance structure on F_{ϑ});

(2) if $\lambda_1 \notin \lambda + \Lambda^{\sharp,r}$*, then* $\mathrm{CT}_B^{\lambda,\lambda_1}(\check{\rho})[2\check{\rho}] = 0$.

3.5.5. In the remainder of this subsection, we fix a splitting of $B \twoheadrightarrow T$ and regard T as a maximal torus in G.

For each $\lambda \in \Lambda$, we write S^λ for the base change of $\mathrm{Gr}_B \to \mathrm{Gr}_T$ to X^λ. Using the locally closed immersion $\mathrm{Gr}_B \to \mathrm{Gr}_G$, we may view S^λ as a locally closed sub-indscheme of Gr_G. In particular, we may form the subschemes $\mathrm{Gr}_G^{\le\lambda} \cap S^{\lambda_1}$, $\mathrm{Gr}_G^{\lambda} \cap S^{\lambda_1} \subset \mathrm{Gr}_G$.

For $\lambda \in \Lambda^{\sharp,+}$ and $\lambda_1 \in \Lambda^\sharp$, the pair $(\mathrm{Gr}_G^\lambda \cap S^{\lambda_1}, \mathscr{G})$ of a scheme equipped with an A-gerbe maps to the pairs $(X, F_\vartheta(\lambda))$ and $(X, F_\vartheta(\lambda_1))$, as induced from the two inclusions in the following diagram:

$$\begin{array}{ccccc} \mathrm{Gr}_G^\lambda & \supset & \mathrm{Gr}_G^\lambda \cap S^{\lambda_1} & \subset & S^{\lambda_1} \\ \downarrow{\scriptstyle p^\lambda} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p^{\lambda_1}} \\ X^\lambda & \cong & X & \cong & X^{\lambda_1} \end{array} \tag{3.29}$$

and the isomorphism of Lemma 3.3.5.

The identifications of $\mathscr{G}$ over $\mathrm{Gr}_G^\lambda \cap S^{\lambda_1}$ with both $p^*F_\vartheta(\lambda)$ and $p^*F_\vartheta(\lambda_1)$ compose into an isomorphism:

$$p^*F_\vartheta(\lambda) \cong p^*F_\vartheta(\lambda_1). \tag{3.30}$$

If $\lambda_1 \in W\lambda$, the identification (3.30) coincides with the one induced from the W-invariance structure of F_ϑ.

Proof of Proposition 3.5.4(1). This part is identical to the non-metaplectic context. Indeed, when $\lambda_1 \in W \cdot \lambda$, the inclusion $\mathrm{Gr}_G^\lambda \cap S^{\lambda_1} \subset \mathrm{Gr}_G^{\le\lambda} \cap S^{\lambda_1}$ is an isomorphism and both schemes are identified with the L_+N-orbit of X^λ (see the proof of [MV07, Theorem 3.2]).

The projection p in (3.29) is thus an affine space bundle of fiber rank $\langle \check{\rho}, \lambda_1 + \lambda \rangle$. Hence the functor $p_! p^*$ is canonically equivalent to the value of $(-\check{\rho})[-2\check{\rho}]$ at $\lambda + \lambda_1$. As $\mathrm{CT}_B^{\lambda,\lambda_1}$ is identified with $p_! p^*$, the desired conclusion follows. □

3.5.6. To prove Proposition 3.5.4(2), we need an additional piece of datum associated to the isomorphism (3.30).

To define it, we briefly return to a more abstract setting: S is any base scheme and A is a finite abelian group, $H \to S$ is a group scheme, and Y is an S-scheme equipped with an H-action.

Suppose that $\mathscr{G}_1$, $\mathscr{G}_2$ are A-gerbes over Y equipped with H-equivariance structures. Let $f : \mathscr{G}_1 \cong \mathscr{G}_2$ be an isomorphism of plain A-gerbes over Y. Consider the diagram formed by the action, respectively projection maps from $H \times Y$ to Y:

$$\begin{array}{ccc} \mathrm{pr}^*(\mathscr{G}_1) & \xrightarrow{\mathrm{pr}^*(f)} & \mathrm{pr}^*(\mathscr{G}_2) \\ \downarrow & & \downarrow \\ \mathrm{act}^*(\mathscr{G}_1) & \xrightarrow{\mathrm{act}^*(f)} & \mathrm{act}^*(\mathscr{G}_2) \end{array} \tag{3.31}$$

The diagram (3.31) needs not commute. The quotient of its upper circuit by its lower circuit defines an A-torsor over $H \times Y$ rigidified along $e \times Y$, which we call the *obstruction of* f *to be* H*-equivariant.*[4]

[4] This terminology should not be taken seriously, as it says nothing about the cocycle condition. The only case of interest for us is when H is a torus, where the obstructions are discrete and the cocycle conditions are automatic.

3.5.7. We shall apply the above construction to the isomorphism (3.30) of A-gerbes over $\mathrm{Gr}_G^\lambda \cap S^{\lambda_1}$, equipped with the T_{ad}-action. Here, T_{ad} is the maximal torus of the adjoint group G_{ad} induced from T, acting by automorphisms of G. (We write Λ_{ad} for the sheaf of cocharacters of T.) The A-gerbes $p^*F_\vartheta(\lambda)$ and $p^*F_\vartheta(\lambda_1)$ are T_{ad}-equivariant, as they are pulled back from X.

Using the isomorphism (3.4) between rigidified A-torsors over T_{ad} and characters $\Lambda_{ad} \to \underline{A}(-1)$, we may describe the obstruction of (3.30) to be T_{ad}-equivariant as a locally constant section:

$$\tau^{\lambda,\lambda_1} \in \underline{\mathrm{Hom}}(\Lambda_{ad}, \underline{A}(-1)) \text{ over } \mathrm{Gr}_G^\lambda \cap S^{\lambda_1}. \tag{3.32}$$

3.5.8. Note that the equality (1.22) shows that the restriction of b to $\Lambda \otimes \Lambda_{sc}$, where $\Lambda_{sc} \subset \Lambda$ is the span of coroots, extends to a bilinear form:

$$\tilde{b} : \Lambda_{ad} \otimes \Lambda_{sc} \to \underline{A}(-1), \quad (\lambda \in \Lambda_{ad}, \alpha \in \Phi) \mapsto Q(\alpha)\langle \lambda, \check{\alpha} \rangle. \tag{3.33}$$

Lemma 3.5.9. *Suppose that $\lambda \in \Lambda^{\sharp,+}$, $\lambda_1 \in \Lambda^\sharp$ are such that $\mathrm{Gr}_G^\lambda \cap S^{\lambda_1} \neq \varnothing$. (This implies $\lambda - \lambda_1 \in \Lambda_{sc}$.) Then the obstruction* (3.32) *of the isomorphism* (3.30) *to be T_{ad}-equivariant is equal to the constant character $\tilde{b}(-, \lambda_1 - \lambda)$.*

Proof. The calculation can be performed over $\bar{k}$-points of X. From now on, we fix a $\bar{k}$-point $\bar{x}$ of X with local uniformizer ϖ. The notations Gr_G^λ and S^{λ_1} now stand for their base changes to $\bar{x}$. The closed immersion $X^\lambda \to \mathrm{Gr}_G^\lambda$ corresponds to the $\bar{k}$-point $\bar{x}^\lambda$ of Gr_G^λ. It lifts to a $\bar{k}$-point ϖ^λ of L(G).

Let $B^- \subset G$ denote the Borel subgroup opposite to B. It has associated $L(N^-)$-orbit $S^{-,\lambda}$ of $\bar{x}^\lambda$ for each $\lambda \in \Lambda$.

Recall that each irreducible component of $\mathrm{Gr}_G^\lambda \cap S^{\lambda_1}$ intersects nontrivially with the "Zastava space" $S^{-,w_0(\lambda)} \cap S^{\lambda_1}$, where $w_0 \in W$ stands for the longest element. Indeed, the fact that $\mathrm{Gr}_G^\lambda \cap S^{-,w_0(\lambda)}$ is dense in Gr_G^λ shows that any irreducible component of $\mathrm{Gr}_G^\lambda \cap S^{\lambda_1}$ is contained in $\overline{S^{-,w_0(\lambda)}} \cap S^{\lambda_1}$. If it belonged to the complement of $S^{-,w_0(\lambda)} \cap S^{\lambda_1}$, it would be of dimension strictly less than $\langle \lambda + \lambda_1, \check{\rho} \rangle$ by the dimension calculation of Zastava spaces ([BFGM02, 5.10]), but $\mathrm{Gr}_G^\lambda \cap S^{\lambda_1}$ is pure of dimension $\langle \lambda + \lambda_1, \check{\rho} \rangle$ ([MV07, Theorem 3.2]).

Let us consider the analogue of (3.29) for the Zastava space:

$$\begin{array}{ccccc} S^{-,\lambda} & \supset & S^{-,\lambda} \cap S^{\lambda_1} & \subset & S^{\lambda_1} \\ \downarrow{\scriptstyle p^\lambda} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p^{\lambda_1}} \\ \bar{x}^\lambda & \cong & \bar{x} & \cong & \bar{x}^{\lambda_1} \end{array} \tag{3.34}$$

and the induced isomorphism of A-gerbes over $S^{-,\lambda} \cap S^{\lambda_1}$:

$$p^*F_\vartheta(\lambda) \cong p^*F_\vartheta(\lambda_1). \tag{3.35}$$

It suffices to prove: *the obstruction of* (3.35) *to be T_{ad}-equivariant is equal to $\tilde{b}(-, \lambda_1 - \lambda)$.* Indeed, the desired statement over $\mathrm{Gr}_G^\lambda \cap S^{\lambda_1}$ will then follow from the equality $\tilde{b}(-, \lambda_1 - \lambda) = \tilde{b}(-, \lambda_1 - w_0(\lambda))$, as $w_0(\lambda) - \lambda \in \Lambda^{\sharp,r}$ is annihilated by Q.

To calculate the obstruction of (3.35) to be T_{ad}-equivariant, we may assume $\lambda_1 = 0$. Indeed, because $\mathscr{G}$ is induced from an $\mathbb{E}_1$-monoidal morphism $L(G) \to B^2(\underline{A})$, pulling back (3.35) along the isomorphism defined by multiplication by ϖ^{λ_1}:

$$\varpi^{\lambda_1} : S^{-,\lambda-\lambda_1} \cap S^0 \cong S^{-,\lambda} \cap S^{\lambda_1}$$

yields the product of the isomorphism $p^*F_\vartheta(\lambda - \lambda_1) \cong p^*F_\vartheta(0)$ with the identity automorphism of $p^*F_\vartheta(\lambda_1)$.

The assumption $\lambda_1 = 0$ forces $\lambda \in \Lambda_{\mathrm{sc}}$, so $\mathrm{S}^{-,\lambda} \cap \mathrm{S}^0$ is contained in the neutral component of Gr_{G}.

Let $\mathrm{G}_{\mathrm{sc}} \to \mathrm{G}$ denote the simply connected form of G, with induced maximal torus $\mathrm{T}_{\mathrm{sc}} \subset \mathrm{G}_{\mathrm{sc}}$. Write $\mathrm{S}^{-,\lambda}_{\mathrm{sc}}$, $\mathrm{S}^0_{\mathrm{sc}}$ for the corresponding orbits in $\mathrm{Gr}_{\mathrm{G}_{\mathrm{sc}}}$ and $p_{\mathrm{sc}} : \mathrm{S}^{-,\lambda}_{\mathrm{sc}} \cap \mathrm{S}^0_{\mathrm{sc}} \to \bar{x}$ the projection. The pullback of (3.35) to $\mathrm{S}^{-,\lambda}_{\mathrm{sc}} \cap \mathrm{S}^0_{\mathrm{sc}}$ is the composition of isomorphisms:

$$(p_{\mathrm{sc}})^* \mathrm{F}_\vartheta(\lambda) \cong \mathscr{G}_{\mathrm{sc}} \cong (p_{\mathrm{sc}})^* \mathrm{F}_\vartheta(0), \tag{3.36}$$

where $\mathscr{G}_{\mathrm{sc}}$ denotes (the restriction of) the A-gerbe over $\mathrm{Gr}_{\mathrm{G}_{\mathrm{sc}}}$ defined by the pullback μ_{sc} of μ along $\mathrm{B}(\mathrm{G}_{\mathrm{sc}}) \to \mathrm{B}(\mathrm{G})$. The two isomorphisms in (3.36) are induced from the inclusion of $\mathrm{S}^{-,\lambda}_{\mathrm{sc}} \cap \mathrm{S}^0_{\mathrm{sc}}$ in $\mathrm{S}^{-,\lambda}_{\mathrm{sc}}$, respectively $\mathrm{S}^0_{\mathrm{sc}}$.

Since the $\bar{k}$-points of $\mathrm{Gr}_{\mathrm{G}_{\mathrm{sc}}}$ map bijectively to those of the neutral component of Gr_{G}, it suffices to calculate the obstruction of (3.36) to be T_{ad}-equivariant. (Note that T_{ad} acts by automorphisms of G_{sc}, as the latter is functorially attached to G.)

We shall now appeal to the canonical T_{ad}-equivariance structure of $\mathscr{G}_{\mathrm{sc}}$. Indeed, the rigidified morphism $\mu_{\mathrm{sc}} : \mathrm{B}(\mathrm{G}_{\mathrm{sc}}) \to \mathrm{B}^4\underline{\mathrm{A}}(1)$ is T_{ad}-equivariant, and the "$(\mathrm{T}_{\mathrm{ad}}, \mathrm{T}_{\mathrm{sc}})$-commutator" of the induced $\mathbb{E}_1$-monoidal morphism $\mathrm{T}_{\mathrm{sc}} \to \mathrm{B}^3\underline{\mathrm{A}}(1)$ is the bi-rigidified morphism:

$$\mathrm{T}_{\mathrm{ad}} \times \mathrm{T}_{\mathrm{sc}} \to \mathrm{B}^2\underline{\mathrm{A}}(1),$$

defined by the pairing $\tilde{b}$ (see [Zha22, §5.5]).

As in the proof of Lemma 3.1.7 (task (2)), the A-gerbe $\mathscr{G}_{\mathrm{sc}} \otimes (p_{\mathrm{sc}})^* \mathrm{F}_\vartheta(\lambda)^{\otimes -1}$ over $\mathrm{S}^{-,\lambda}_{\mathrm{sc}}$ descends to the rigidified A-gerbe $\Psi^{\tilde{b}(-,\lambda)}$ over $\mathrm{B}_{\bar{x}^\lambda}(\mathrm{T}_{\mathrm{ad}})$. Similarly, the A-gerbe $\mathscr{G}_{\mathrm{sc}} \otimes (p_{\mathrm{sc}})^* \mathrm{F}_\vartheta(0)^{\otimes -1}$ over $\mathrm{S}^0_{\mathrm{sc}}$ descends to the (trivial) rigidified A-gerbe $\Psi^{\tilde{b}(-,0)}$ over $\mathrm{B}_{\bar{x}^0}(\mathrm{T}_{\mathrm{ad}})$. The obstruction of (3.36) to be T_{ad}-equivariant is thus the difference of obstructions:

$$-\tilde{b}(-,0) + \tilde{b}(-,\lambda) = \tilde{b}(-,\lambda).$$

This establishes the desired equality. □

Remark 3.5.10. The proof of Lemma 3.5.9 also establishes its variant where $\mathrm{Gr}^\lambda_{\mathrm{G}} \cap \mathrm{S}^{\lambda_1}$ is replaced by the Zastava space $\mathrm{S}^{-,\lambda} \cap \mathrm{S}^{\lambda_1}$.

Proof of Proposition 3.5.4(2). The λ-summand of $\mathrm{Sat}_{\mathrm{G},\mathscr{G}}$ consists of intermediate extensions of $\mathscr{G}$-twisted E-local systems along $\mathrm{Gr}^\lambda_{\mathrm{G}} \subset \mathrm{Gr}^{\le\lambda}_{\mathrm{G}}$, so their restrictions to any boundary stratum $\mathrm{Gr}^{\lambda_2}_{\mathrm{G}}$ lie in perverse cohomological degrees $\le -\langle 2\check{\rho}, \lambda_2\rangle - 1$. The t-exactness of $\mathrm{CT}^{\lambda,\lambda_1}(\check{\rho})[2\check{\rho}]$ implies that only the open stratum $\mathrm{Gr}^\lambda_{\mathrm{G}} \cap \mathrm{S}^{\lambda_1}$ contributes. In other words, it is isomorphic to the functor:

$$p_! p^*(\langle \check{\rho}, \lambda + \lambda_1\rangle)[\langle 2\check{\rho}, \lambda + \lambda_1\rangle] : \mathrm{Lis}_{\mathrm{F}_\vartheta(\lambda)}(\mathrm{X}) \to \mathrm{Lis}_{\mathrm{F}_\vartheta(\lambda_1)}(\mathrm{X}), \tag{3.37}$$

defined by the isomorphism (3.30) of A-gerbes.

The vanishing of (3.37) can be verified over $\bar{k}$-points. From now on, we fix a $\bar{k}$-point $\bar{x}$ of X and choose trivializations of the A-gerbes $\mathrm{F}_\vartheta(\lambda)_{\bar{x}}$ and $\mathrm{F}_\vartheta(\lambda_1)_{\bar{x}}$ over $\bar{x}$.

The isomorphism (3.30) thus defines an A-torsor τ over $\mathrm{Gr}^\lambda_{\mathrm{G}} \cap \mathrm{S}^{\lambda_1}$, and the image of $\underline{\mathrm{E}}$ under (3.37) is isomorphic to the vector space:

$$\mathrm{H}^{\langle 2\check{\rho}, \lambda+\lambda_1\rangle}_c(\mathrm{Gr}^\lambda_{\mathrm{G}} \cap \mathrm{S}^{\lambda_1}, \mathscr{L}(\langle \check{\rho}, \lambda + \lambda_1\rangle)), \tag{3.38}$$

where $\mathscr{L}$ is the rank-1 E-local system induced from τ along $\mathrm{A} \subset \mathrm{E}^\times$.

If $\lambda_1 - \lambda \notin \Lambda_{\mathrm{sc}}$, then $\mathrm{Gr}^\lambda_{\mathrm{G}} \cap \mathrm{S}^{\lambda_1} = \varnothing$ and (3.38) clearly vanishes.

Suppose that $\lambda_1 - \lambda \in \Lambda_{\mathrm{sc}}$. We write $\lambda_1 - \lambda = \sum_\alpha d_\alpha \alpha$ for $\alpha \in \Delta$ and $d_\alpha \in \mathbb{Z}$. The hypothesis $\lambda_1 - \lambda \notin \Lambda^{\sharp,r}$ means that d_α is *indivisible* by $\mathrm{ord}(\mathrm{Q}(\alpha))$ for some $\alpha \in \Delta$.

By Lemma 3.5.9, the A-torsor τ is T_{ad}-equivariant against the multiplicative A-torsor $\Psi^{\tilde{b}(-,\lambda_1-\lambda)}$. Using Lemma 1.3.5, we see that (3.38) vanishes as long as $\mathrm{H}^0(\mathrm{T}_{\mathrm{ad}}, \Psi^{\tilde{b}(-,\lambda_1-\lambda)}) =$

0. The latter vanishing follows from the hypothesis $\lambda_1 - \lambda \notin \Lambda^{\sharp,r}$, as the pullback of $\Psi^{\tilde{b}(-,\lambda_1-\lambda)}$ along the fundamental coweight $\delta_\alpha : \mathbb{G}_m \to \mathrm{T}_{\mathrm{ad}}$ dual to $\check{\alpha} \in \check{\Delta}$ yields $\Psi^{d_\alpha \mathrm{Q}(\alpha)}$. □

3.6. Constant terms: fiber functor.

3.6.1. We return to the context of §3.4.1. The goal of this subsection is to use the constant term functor associated to B to construct a "fiber functor" for $\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}$.

Our first task is to record a corollary of Proposition 3.5.4(1), which concerns those properties reflected by the constant term functor.

Lemma 3.6.2. *For any nonempty finite set* I, *the functor* $\mathrm{CT}^{\mathrm{I}}_{\mathrm{P}}(\check{\rho}_{\mathrm{P}})[2\check{\rho}_{\mathrm{P}}]$ (3.24) *satisfies the following properties:*

(1) it is conservative;
(2) an object of $\mathrm{Shv}_{\mathscr{G}^{\mathrm{I}}}(\mathrm{Hec}^{\mathrm{I}}_{\mathrm{G}})$ *is universally locally acyclic relative to* X^{I} *if and only if its image is.*

Proof. Since the functors (3.24) are compatible with compositions (Remark 3.4.5), we may assume P = B.

Statement (1) is reduced to its analogue over $\bar{k}$-points of X^{I}, thus to the case $\mathrm{I} = \{1\}$ after possibly replacing G by a product of copies of G. Then it follows from the decomposition (3.21) and the special case of Proposition 3.5.4(1) for $\lambda_1 = \lambda \in \Lambda^{\sharp,+}$.

Statement (2) follows from the criterion [HS21, Theorem 4.4(iv)] of universal local acyclicity and the conservativity of $\mathrm{CT}^{\mathrm{I}}_{\mathrm{B}}(\check{\rho})[2\check{\rho}]$. □

3.6.3. Our second task is to construct a decomposition of E-linear abelian categories:

$$ {}^{+}\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}} \cong \bigoplus_{\lambda^{\mathrm{I}} \in (\hat{\mathrm{Z}}_{\mathrm{H}})^{\mathrm{I}}} {}^{+}\mathrm{Sat}^{\mathrm{I},\lambda^{\mathrm{I}}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}, \tag{3.39} $$

which is compatible with the tensor structure on ${}^{+}\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}$, *i.e.*

(1) the unit $e_!(\underline{\mathrm{E}})$ belongs to ${}^{+}\mathrm{Sat}^{\mathrm{I},0}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}$;
(2) the monoidal product of $\mathscr{F}_1 \in {}^{+}\mathrm{Sat}^{\mathrm{I},\lambda^{\mathrm{I}}_1}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}$ and $\mathscr{F}_2 \in {}^{+}\mathrm{Sat}^{\mathrm{I},\lambda^{\mathrm{I}}_2}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}$ belongs to ${}^{+}\mathrm{Sat}^{\mathrm{I},\lambda^{\mathrm{I}}_1+\lambda^{\mathrm{I}}_2}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}$.

Furthermore, the decomposition (3.39) is of étale local nature over X^{I}.

Construction of (3.39). Let $\pi : \Lambda^{\sharp} \twoheadrightarrow \hat{\mathrm{Z}}_{\mathrm{H}}$ denote the projection map. (Recall that $\hat{\mathrm{Z}}_{\mathrm{H}}$ is the quotient of $\Lambda^{\sharp}$ by $\Lambda^{\sharp,r}$.) We proceed in increasing generality.

Case: $\mathrm{I} = \{1\}$. For each $\lambda \in \hat{\mathrm{Z}}_{\mathrm{H}}$, we define the direct summand:

$$ {}^{+}\mathrm{Sat}^{\{1\},\lambda}_{\mathrm{G},\mathscr{G}^{\{1\}}} := \bigoplus_{\substack{\lambda^{+} \in \Lambda^{\sharp,+} \\ \pi(\lambda^{+}) = \lambda}} \mathrm{Lis}_{\mathrm{F}_{\vartheta}(\lambda^{+})}(\mathrm{X}) \subset {}^{+}\mathrm{Sat}^{\{1\}}_{\mathrm{G},\mathscr{G}^{\{1\}}}, \tag{3.40} $$

according to the decomposition (3.21).

It is clear that $e_!(\underline{\mathrm{E}})$ belongs to ${}^{+}\mathrm{Sat}^{\{1\},0}_{\mathrm{G},\mathscr{G}^{\{1\}}}$.

To prove the compatibility with monoidal product, we note that by Proposition 3.5.4, the summand (3.40) consists precisely of objects in ${}^{+}\mathrm{Sat}^{\{1\}}_{\mathrm{G},\mathscr{G}^{\{1\}}}$ whose images under $\mathrm{CT}^{\{1\}}_{\mathrm{B}}(\check{\rho})[2\check{\rho}]$ are supported on the strata X^{λ_1}, for $\lambda_1 \in \Lambda^{\sharp}$ with $\pi(\lambda_1) = \lambda$. However, for the torus T, the decomposition (3.21) is compatible with the monoidal product. Thus the same holds for the decomposition of ${}^{+}\mathrm{Sat}^{\{1\}}_{\mathrm{G},\mathscr{G}^{\{1\}}}$ defined by the summands (3.40).

Case: disjoint locus. For a nonempty finite set I, we consider the identity map $p = \mathrm{id}_{\mathrm{I}}$. The open subscheme $\mathrm{X}^{p} \subset \mathrm{X}^{\mathrm{I}}$ is the pairwise disjoint locus.

For each $\lambda^{\mathrm{I}} \in (\hat{\mathrm{Z}}_{\mathrm{H}})^{\mathrm{I}}$, we set:

$$ {}^{+}\mathrm{Sat}^{p,\lambda^{\mathrm{I}}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}} \subset {}^{+}\mathrm{Sat}^{p}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}} \tag{3.41} $$

to be the full subcategory consisting of objects whose images under $\mathrm{CT}^{\mathrm{I}}_{\mathrm{B}}(\check{\rho})[2\check{\rho}]$ are supported on the strata $\mathrm{X}^{\lambda^{\mathrm{I}}_1}$, for $\lambda^{\mathrm{I}}_1 \in (\Lambda^{\sharp})^{\mathrm{I}}$ with $\pi(\lambda^{\mathrm{I}}_1) = \lambda^{\mathrm{I}}$.

The fact that the full subcategories (3.41) induce a direct sum decomposition of ${}^{+}\mathrm{Sat}^{p}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}$ compatible with its tensor structure follows from the case for $\mathrm{I} = \{1\}$.

Case: general. Let I, p, λ^{I} be as above. We set:

$$ {}^{+}\mathrm{Sat}^{\mathrm{I},\lambda^{\mathrm{I}}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}} \subset {}^{+}\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}} \tag{3.42} $$

to be the full subcategory consisting of objects whose restrictions along $\mathrm{X}^{p} \subset \mathrm{X}^{\mathrm{I}}$ belong to the full subcategory (3.41).

The fact that (3.42) induces a direct sum decomposition of ${}^{+}\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}$ follows from the closure of the full subcategory ${}^{+}\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}} \subset {}^{+}\mathrm{Sat}^{p}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}$ under direct summands. Its compatibility with tensor structure follows from the case for the disjoint locus. □

Remark 3.6.4. For $\mathrm{I} = \{1\}$, the decomposition (3.39) coarsens the decomposition (3.21). However, the latter decomposition is *incompatible* with the tensor structure unless $\mathrm{G} = \mathrm{T}$ (*c.f.* Remark 3.3.8).

3.6.5. Using the decomposition (3.39), being of étale local nature over X^{I}, we may twist the tensor category ${}^{+}\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}$ by the $\mathbb{E}_\infty$-monoidal morphism negative to $\mathrm{F}^{\mathrm{I}}_{\vartheta}$ (*i.e.* the formation (1.32) applied to $\mathrm{F}^{\otimes -1}_{\vartheta}$):

$$ (\mathrm{F}^{\mathrm{I}}_{\vartheta})^{\otimes -1} : (\hat{\mathrm{Z}}_{\mathrm{H}})^{\mathrm{I}} \to \mathrm{B}^2_{\mathrm{X}^{\mathrm{I}}}(\mathrm{A}), \quad (\lambda_i)_{i\in\mathrm{I}} \mapsto \boxtimes_{i\in\mathrm{I}} \mathrm{F}_{\vartheta}(\lambda_i)^{\otimes -1}. $$

This process yields a tensor category $({}^{+}\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}})_{(\mathrm{F}^{\mathrm{I}}_{\vartheta})^{\otimes -1}}$.

Combining $\mathrm{CT}^{\mathrm{I}}_{\mathrm{B}}(\check{\rho})[2\check{\rho}]$ with the geometric Satake equivalence for split tori (§3.2.5), we find a tensor functor:

$$ \begin{aligned} \omega^{\mathrm{I}} : ({}^{+}\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}})_{(\mathrm{F}^{\mathrm{I}}_{\vartheta})^{\otimes -1}} &\xrightarrow{\mathrm{CT}^{\mathrm{I}}_{\mathrm{B}}(\check{\rho})[2\check{\rho}]} (\mathrm{Sat}^{\mathrm{I}}_{\mathrm{T},\mathscr{G}^{\mathrm{I}}_{\mathrm{T}}})_{(\mathrm{F}^{\mathrm{I}}_{\vartheta})^{\otimes -1}} \\ &\cong (\mathrm{Rep}^{\mathrm{I}}_{\mathrm{T}^{\mathrm{I}}_{\mathrm{H}},\mathrm{F}^{\mathrm{I}}_{\vartheta}})_{(\mathrm{F}^{\mathrm{I}}_{\vartheta})^{\otimes -1}} \cong \mathrm{Rep}^{\mathrm{I}}_{\mathrm{T}^{\mathrm{I}}_{\mathrm{H}}} \to \mathrm{Lis}(\mathrm{X}^{\mathrm{I}}), \end{aligned} \tag{3.43} $$

where $\mathrm{T}_{\mathrm{H}} \subset \mathrm{H}$ is the maximal torus (with character lattice $\Lambda^{\sharp}$), and the last functor is the one forgetting the $\mathrm{T}^{\mathrm{I}}_{\mathrm{H}}$-action.

Remark 3.6.6. The tensor functor ω^{I} satisfies the following additional properties:

(1) it is conservative (Lemma 3.6.2(1));
(2) it is exact (Proposition 3.4.4);
(3) it is independent of the choice of B (Remark 3.4.6);

3.7. Tannakian reconstruction.

3.7.1. Consider the 2-category whose objects are tensor (*i.e.* E-linear symmetric monoidal abelian) categories, whose morphisms are tensor functors, and whose 2-morphisms are natural transformations compatible with the tensor structures.

A tensor category is called *rigid* if all of its objects are dualizable. Note that in a rigid tensor category, the monoidal product with any object is exact.

3.7.2. Let **A** be a rigid tensor category. The following categories are related by a pair of adjoint functors:

(1) commutative Hopf algebras in $\mathrm{Ind}(\mathbf{A})$;
(2) pairs $(\mathbf{C},\omega)$ where $\mathbf{C}$ is a rigid tensor category under $\mathbf{A}$ and $\omega : \mathbf{C} \to \mathbf{A}$ is a conservative, exact, $\mathbf{A}$-linear tensor functor.

The functor (1) $\Rightarrow$ (2) associates to a Hopf algebra $\mathrm{A} \in \mathrm{Ind}(\mathbf{A})$ its tensor category of comodules $\mathbf{C} := \mathrm{Comod}_{\mathrm{A}}(\mathbf{A})$ on objects of $\mathbf{A}$ and the forgetful functor ω.

Its left adjoint (2) $\Rightarrow$ (1) associates to a pair $(\mathbf{C},\omega)$ the object $\mathrm{Ind}(\omega)\cdot \mathrm{G}(1) \in \mathrm{Ind}(\mathbf{A})$, where $1 \in \mathbf{A}$ is the tensor unit. Here, the ind-extension $\mathrm{Ind}(\omega)$ preserves all colimits, and thus admits a right adjoint G. Since $\mathrm{Ind}(\omega)$ is a tensor functor, G preserves commutative algebras. The coalgebra structure is induced from the comonad $\mathrm{Ind}(\omega)\cdot \mathrm{G}$. The existence of inverse follows from the rigidity of $\mathbf{C}$.

Lemma 3.7.3. *The adjunction in §3.7.2 is an equivalence.*

Proof. We show that the counit and unit are isomorphisms.

Counit. Let $\mathrm{A} \in \mathbf{A}$ be a bi-algebra. The right adjoint G is given by $(-)\otimes \mathrm{A}$, where A is viewed as an A-comodule by its coalgebra structure. The counit map $\mathrm{Ind}(\omega)\circ \mathrm{G}(1) \to \mathrm{A}$ is thus an isomorphism.

Unit. Let $(\mathbf{C},\omega)$ be a pair as in §3.7.2(2). The functor $\mathrm{Ind}(\omega)$ remains conservative, so the adjunction $(\mathrm{Ind}(\omega),\mathrm{G})$ satisfies the hypothesis of the Barr–Beck theorem, *i.e.* it is comonadic. We see that $\mathrm{Ind}(\mathbf{C})$ is equivalent to the category of $(\mathrm{Ind}(\omega)\cdot \mathrm{G})$-comodules in $\mathrm{Ind}(\mathbf{A})$, compatibly with the forgetful functors.

Since $\mathrm{Ind}(\omega)$ is $\mathrm{Ind}(\mathbf{A})$-linear, so is the functor G. This shows that the comonad $\mathrm{Ind}(\omega)\cdot\mathrm{G}$ is identified with tensoring by the coalgebra $\mathrm{A} := \mathrm{Ind}(\omega)\cdot \mathrm{G}(1)$.

Finally, we must show that $\mathbf{C} \subset \mathrm{Ind}(\mathbf{C})$ coincides with the full subcategory of objects whose image under $\mathrm{Ind}(\omega)$ belongs to $\mathbf{A}$. Since $\mathbf{C}$ (resp. $\mathbf{A}$) is abelian, hence idempotent-complete, it is the full subcategory of compact objects of $\mathrm{Ind}(\mathbf{C})$ (resp. $\mathrm{Ind}(\mathbf{A})$). We conclude using the observation that $\mathrm{Ind}(\omega)$ reflects compactness. ☐

3.7.4. For a nonempty finite set I, the tensor category ${}^{+}\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}$ admits a $\mathrm{Lis}(\mathrm{X}^{\mathrm{I}})$-structure supplied by $e_!$, for $e : \mathrm{X}^{\mathrm{I}} \to \mathrm{Hec}^{\mathrm{I}}_{\mathrm{G}}$ being the unit section.

Moreoever, ${}^{+}\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}$ is rigid. Indeed, by the definition of the fusion product, we can check the dualizability of an object in ${}^{+}\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}$ over the pairwise disjoint locus of X^{I}. There, the statement reduces to $\bar{k}$-points and the case $\mathrm{I} = \{1\}$, where it follows from the argument of [Zhu17, Theorem 5.2.9].

The same assertions are inherited by the twisted tensor category $({}^{+}\mathrm{Sat}^{\mathrm{I}}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}})_{(\mathrm{F}^{\mathrm{I}}_{\vartheta})^{\otimes -1}}$, so Lemma 3.7.3 produces a commutative Hopf algebra $\mathrm{A}^{\mathrm{I}} \in \mathrm{Ind}(\mathrm{Lis}(\mathrm{X}^{\mathrm{I}}))$.

Proposition 3.7.5. *There is a canonical isomorphism of Hopf algebras:*

$$\mathrm{A}^{\mathrm{I}} \cong \mathscr{O}_{\mathrm{H}^{\mathrm{I}}} \quad \in \mathrm{Ind}(\mathrm{Lis}(\mathrm{X}^{\mathrm{I}})). \tag{3.44}$$

3.7.6. Let H_1 denote the locally constant étale sheaf over X of affine group schemes over E corresponding to $\mathrm{A}^{\{1\}}$.

The isomorphism (3.44) for $\mathrm{G} = \mathrm{T}$ is constructed in §3.2.5. The composition of all but the last functor in the definition of $\omega^{\{1\}}$ (3.43) yields a homomorphism from the maximal torus $\mathrm{T}_{\mathrm{H}} \subset \mathrm{H}$ to H_1.

The isomorphism $\mathrm{H}_1 \cong \mathrm{H}$ supplied by (3.44) will extend the identity map on T_{H}.

Lemma 3.7.7. *The following statements hold:*

(1) H_1 *is a sheaf of reductive group schemes;*
(2) the map $\mathrm{T}_{\mathrm{H}} \to \mathrm{H}_1$ *realizes* T_{H} *as a maximal torus of* H_1.

Proof. Both statements may be verified over $\bar{k}$-points of X. We shall now fix a $\bar{k}$-point $\bar{x}$ and write $H_{1,\bar{x}}$ (resp. $T_{H,\bar{x}}$) for the stalk of H_1 (resp. T_H) at $\bar{x}$.

By Proposition 3.3.7, the underlying E-linear abelian category of finite-dimensional representations of $H_{1,\bar{x}}$ decomposes as a sum of the category of finite-dimensional E-vector spaces, indexed by $\Lambda^{\sharp,+}$. We write $\mathscr{E}^{\lambda} \in \mathrm{Rep}_{H_{1,\bar{x}}}$ for the object corresponding to the 1-dimensional E-vector space E and index $\lambda \in \Lambda^{\sharp,+}$.

Observe that $H_{1,\bar{x}}$ is of finite type. Indeed, $\Lambda^{\sharp,+}$ is finitely generated as a monoid and $\mathscr{E}^{\lambda_1+\lambda_2}$ appears as a summand of $\mathscr{E}^{\lambda_1} \otimes \mathscr{E}^{\lambda_2}$ by Proposition 3.5.4(1). Thus $\mathrm{Rep}_{H_{1,\bar{x}}}$ has a finite number of tensor generators, so [DM82, Proposition 2.20] applies.

Next, $H_{1,\bar{x}}$ is connected because it does not have nontrivial finite tensor subcategories ([DM82, Corollary 2.22]).

Finally, $H_{1,\bar{x}}$ is reductive because $\mathrm{Rep}_{H_{1,\bar{x}}}$ is semisimple ([DM82, Proposition 2.23]) according to its aforementioned decomposition. This proves statement (1).

By Proposition 3.5.4(1), any object of $\mathrm{Rep}_{T_{H,\bar{x}}}$ is a subobject of an object coming from $\mathrm{Rep}_{H_{1,\bar{x}}}$. The morphism $T_{H,\bar{x}} \to H_{1,\bar{x}}$ is thus a closed immersion by [DM82, Proposition 2.21(b)]. The argument of [Zhu17, Lemma 5.3.17], substituting Proposition 3.5.4 for Theorem 5.3.9 in *loc.cit.*, shows that $T_{H,\bar{x}}$ is a maximal torus. □

3.7.8. Let us fix a $\bar{k}$-point $\bar{x}$ of X. We shall upgrade the maximal torus $T_{H,\bar{x}} \subset H_{1,\bar{x}}$ into a pinning of the reductive group scheme $H_{1,\bar{x}}$.

Denote by $2\rho_H \in \check{\Lambda}^{\sharp}$ the sum of positive coroots of $H_{\bar{x}}$. As in [BR18, §9.2], we choose a Borel subgroup $B_{H_1,\bar{x}} \subset H_{1,\bar{x}}$ containing $T_{H,\bar{x}}$ such that $2\rho_H$ is a dominant cocharacter with respect to $B_{H_1,\bar{x}}$. This choice has the property that the dominant *characters* of $T_{H,\bar{x}}$ it defines agree with those defined by $B_{H,\bar{x}} \subset H_{\bar{x}}$ (see [BR18, Lemma 9.5]).

We record the data which have been constructed:

$$\begin{array}{ccccc} T_{H,\bar{x}} & \subset & B_{H_1,\bar{x}} & \subset & H_{1,\bar{x}} \\ \downarrow{\scriptstyle\cong} & & & & \\ T_{H,\bar{x}} & \subset & B_{H,\bar{x}} & \subset & H_{\bar{x}} \end{array} \tag{3.45}$$

Lemma 3.7.9. *The two rows of* (3.45) *induce the same based root data on the character lattice of* $T_{H,\bar{x}}$.

Proof. The construction of $B_{H_1,\bar{x}}$ being compatible with the constant term functor (3.24), we reduce to the case where G is of semisimple rank one. There, we must show that $B_{H_1,\bar{x}} \subset H_{1,\bar{x}}$ has a unique simple root given by $\mathrm{ord}(Q(\alpha))\alpha \in \Lambda^{\sharp}$, and its associated coroot is given by $\mathrm{ord}(Q(\alpha))^{-1}\check{\alpha}$.

To identify the simple roots, it suffices to show that the $T_{H,\bar{x}}$-weights of the simple object $\mathscr{E}^{\lambda} \in \mathrm{Rep}_{H_{1,\bar{x}}}$ for $\lambda \in \Lambda^{\sharp,+}$ (notations as in the proof of Lemma 3.7.7) is given by:

$$\lambda - d \cdot \mathrm{ord}(Q(\alpha))\alpha, \quad 0 \le d \le \mathrm{ord}(Q(\alpha))^{-1}\langle\check{\alpha}, \lambda\rangle. \tag{3.46}$$

We argue as in [FL10, §4.4]. Proposition 3.5.4(2) implies that the $T_{H,\bar{x}}$-weights of $\mathscr{E}^{\lambda}$ are contained in the set (3.46). To show that they exhaust the latter, it suffices to prove that the A-torsor τ^{λ,λ_1} over $\mathrm{Gr}^{\lambda}_{G,\bar{x}} \cap S^{\lambda_1}_{\bar{x}}$ defined by (3.30) and arbitrary trivializations of $F_{\vartheta}(\lambda)_{\bar{x}}$, $F_{\vartheta}(\lambda_1)_{\bar{x}}$ is (non-canonically) trivial whenever λ_1 belongs to (3.46).

The statement for the extremal cases $\lambda_1 = \lambda$, $\lambda_1 = s_{\alpha}(\lambda)$ follows from Proposition 3.5.4(1). It remains to treat the intermediate cases:

$$\lambda_1 = \lambda - d \cdot \mathrm{ord}(Q(\alpha))\alpha, \quad 1 \le d \le \mathrm{ord}(Q(\alpha))^{-1}\langle\check{\alpha}, \lambda\rangle - 1.$$

Write $a := \langle\check{\alpha}, \lambda\rangle$ and $a_1 := \langle\check{\alpha}, \lambda_1\rangle$, so $a > 0$ and $|a_1| < a$. Identifying N with $\mathbb{G}_a$, the $\mathrm{L}(\mathrm{N})_{\bar{x}}$-action on $\bar{x}^{\lambda_1}$ induces an isomorphism:

$$\mathbb{A}^{a+a_1} \cong \varpi^{-a+a_1}\mathrm{L}_+(\mathbb{G}_a)_{\bar{x}}/\varpi^{2a_1}\mathrm{L}_+(\mathbb{G}_a)_{\bar{x}} \cong \mathrm{Gr}^{\le\lambda}_{\mathrm{G},\bar{x}} \cap \mathrm{S}^{\lambda_1}_{\bar{x}}, \tag{3.47}$$

Under (3.47), the open subscheme $\mathrm{Gr}^{\lambda}_{\mathrm{G},\bar{x}} \cap \mathrm{S}^{\lambda_1}_{\bar{x}}$ corresponds to the locus with invertible leading coefficient in $\varpi^{-a+a_1}\mathrm{L}_+(\mathbb{G}_a)_{\bar{x}}$, hence to $\mathbb{G}_m \times \mathbb{A}^{a+a_1-1} \subset \mathbb{A}^{a+a_1}$. The projection of $\mathrm{Gr}^{\lambda}_{\mathrm{G},\bar{x}} \cap \mathrm{S}^{\lambda_1}_{\bar{x}}$ onto $\mathbb{G}_m$ intertwines the T_{ad}-action with $\mathbb{G}_m$-multiplication.

The A-torsor τ^{λ,λ_1} canonically descends to $\mathbb{G}_m$, where it is (non-canonically) trivial because τ^{λ,λ_1} is T_{ad}-equivariant by Lemma 3.5.9.

The identification of simple roots of $\mathrm{B}_{\mathrm{H}_1,\bar{x}} \subset \mathrm{H}_{1,\bar{x}}$ being complete, the identification of simple coroots follows, because $\mathrm{ord}(\mathrm{Q}(\alpha))^{-1}\check{\alpha}$ is the unique cocharacter pairing non-negatively with all dominant characters of $\mathrm{T}_{\mathrm{H},\bar{x}}$ and yields 2 when paired with $\mathrm{ord}(\mathrm{Q}(\alpha))\alpha$. □

3.7.10. Let us now upgrade the top row of (3.45) into a pinning on $\mathrm{H}_{1,\bar{x}}$.

Construction. The decomposition (3.39) gives rise to two tensor functors:

$$\mathrm{Rep}^0_{\mathrm{H}_1,\bar{x}} \subset \mathrm{Rep}_{\mathrm{H}_1,\bar{x}} \to \bigoplus_{\lambda\in\hat{\mathrm{Z}}_{\mathrm{H}}} \mathrm{Mod}_{\mathrm{E}}, \tag{3.48}$$

where $\mathrm{Rep}^0_{\mathrm{H}_1,\bar{x}}$ denotes the summand corresponding to $0 \in \hat{\mathrm{Z}}_{\mathrm{H}}$, and the second functor is the decomposition of $\omega^{\{1\}}$ according to $\hat{\mathrm{Z}}_{\mathrm{H}}$-weights.

Both functors in (3.48) commute with fiber functors to $\mathrm{Mod}_{\mathrm{E}}$. Thus they define morphisms of affine groups schemes over E:

$$1 \to \mathrm{Z}_{\mathrm{H},\bar{x}} \to \mathrm{H}_{1,\bar{x}} \to \mathrm{H}^0_{1,\bar{x}} \to 1. \tag{3.49}$$

The criterion [EHS07, Theorem A.1(iii)] shows that (3.49) is a short exact sequence. The identification of root data (Lemma 3.7.9) shows that $\mathrm{H}^0_{1,\bar{x}}$ is the adjoint group of $\mathrm{H}_{1,\bar{x}}$. It is equipped with an induced maximal torus and a Borel subgroup:

$$\mathrm{T}^0_{\mathrm{H}_1,\bar{x}} \subset \mathrm{B}^0_{\mathrm{H}_1,\bar{x}} \subset \mathrm{H}^0_{1,\bar{x}}. \tag{3.50}$$

Suppose first that G is of semisimple rank one. Then the same holds for $\mathrm{H}_{1,\bar{x}}$, and $\mathrm{H}^0_{1,\bar{x}}$ has the root data of PGL_2. Any two isomorphisms $\mathrm{H}^0_{1,\bar{x}} \cong \mathrm{PGL}_2$ matching (3.50) with the standard triple $\mathbb{G}_m \subset \mathrm{B} \subset \mathrm{PGL}_2$ differ by the inner automorphism of PGL_2 defined by a unique element $a \in \mathbb{G}_m$. The adjoint action of PGL_2 on the vector space $\mathrm{E}^{\oplus 3}$ differs from its twist by any nontrivial element $a \in \mathbb{G}_m$.

Consider the simple object $\mathscr{E}^{\mathrm{ord}(\mathrm{Q}(\alpha))\alpha} \in \mathrm{Rep}^0_{\mathrm{H}_1,\bar{x}}$. Its image under $\omega^{\{1\}}$ is canonically equivalent to $\mathrm{E}^{\oplus 3}$, using the Tate twist introduced in our constant term functor. Thus there is a unique isomorphism between the triple (3.50) and the standard triple $\mathbb{G}_m \subset \mathrm{B} \subset \mathrm{PGL}_2$ under which $\mathscr{E}^{\mathrm{ord}(\mathrm{Q}(\alpha))\alpha}$ corresponds to the adjoint action of PGL_2 on $\mathrm{E}^{\oplus 3}$.

Under this isomorphism, the pinning of PGL_2 transfers to $\mathrm{H}^0_{1,\bar{x}}$. Since $\mathrm{H}_{1,\bar{x}} \to \mathrm{H}^0_{1,\bar{x}}$ is an isomorphism on root subgroups, $\mathrm{H}_{1,\bar{x}}$ inherits a pinning.

For a general reductive group scheme G, the pinning on $\mathrm{H}_{1,\bar{x}}$ is constructed from the constant term functors and the semisimple rank one case. □

3.7.11. Combining Lemma 3.7.9 and the construction in §3.7.10 of a pinning on $\mathrm{H}_{1,\bar{x}}$, we obtain a *canonical* isomorphism of pinned reductive group schemes over E:

$$\mathrm{H}_{1,\bar{x}} \cong \mathrm{H}_{\bar{x}}, \tag{3.51}$$

for every $\bar{k}$-point $\bar{x}$ of X.

Finally, we shall lift (3.51) to the desired isomorphism (3.44).

Proof of Proposition 3.7.5. We construct the isomorphism (3.44) in increasing generality.

Case: $I = \{1\}$. Claim: H_1 *is constant as an étale sheaf over* X. Once this claim is established, the desired isomorphism $H_1 \cong H$ is supplied by (3.51) at any $\bar{k}$-point $\bar{x}$.

To prove that H_1 is constant, we may assume that X is connected with a fixed $\bar{k}$-point $\bar{x}$. Then H_1 is recovered from its fiber $H_{1,\bar{x}}$ equipped with the $\pi_1(X,\bar{x})$-action. It remains to show that this $\pi_1(X,\bar{x})$-action is trivial.

Since the tensor functor $\mathrm{Rep}_{H_{1,\bar{x}}} \to \mathrm{Rep}_{T_{H,\bar{x}}}$ arises as the fiber at $\bar{x}$ of a tensor functor whose target $\underline{\mathrm{Rep}}^{\{1\}}_{T_H}$ is constant, the $\pi_1(X,\bar{x})$-action on the maximal torus $T_{H,\bar{x}} \subset H_{1,\bar{x}}$ is trivial. This shows that $\pi_1(X,\bar{x})$ acts trivially on the based root data of $H_{1,\bar{x}}$.

It remains to show that the $\pi_1(X,\bar{x})$-action preserves the pinning on $H_{1,\bar{x}}$ constructed in §3.7.10. We may do so under the additional assumption that G is of semisimple rank one, and it suffices to show that $\pi_1(X,\bar{x})$ acts trivially on the adjoint group $H^0_{1,\bar{x}}$.

By functoriality of the intermediate extension, the simple object $\mathscr{E}^{\mathrm{ord}(Q(\alpha))\alpha} \in \mathrm{Rep}^0_{H_{1,\bar{x}}}$ is $\pi_1(X,\bar{x})$-equivariant compatibly with the fiber functor. Thus the corresponding action of $H^0_{1,\bar{x}}$ on $E^{\oplus 3}$ is $\pi_1(X,\bar{x})$-equivariant. However, the induced homomorphism $H^0_{1,\bar{x}} \to GL(E^{\oplus 3})$ is injective, as it coincides with the adjoint representation of PGL_2. It follows that $\pi_1(X,\bar{x})$ acts trivially on $H^0_{1,\bar{x}}$.

Case: general. Let I be any nonempty finite set. The external fusion product of $\mathrm{Sat}^I_{G,\mathscr{G}^I}$ induces a morphism of Hopf algebras:

$$\boxtimes_{i\in I} A^{\{i\}} \to A^I \quad \in \mathrm{Ind}(\mathrm{Lis}(X^I)). \tag{3.52}$$

It suffices to prove that (3.52) is an isomorphism.

This assertion can be proved over the pairwise disjoint locus in X^I and furthermore over any $\bar{k}$-point $\bar{x}$. There, it follows the compatibility between the isomorphism (3.51) and finite product of reductive groups. □

3.7.12. We now prove the geometric Satake equivalence (Theorem 2.4.4) for (G,μ).

Construction of (2.23). Suppose first that G is split.

For any nonempty finite set I, the tensor category $({}^{+}\mathrm{Sat}^I_{G,\mathscr{G}^I})_{(F^I_\vartheta)^{\otimes -1}}$ is identified with $\mathrm{Rep}^I_{H^I}$ by Proposition 3.7.5.

This identification being of étale local nature over X^I, we obtain the desired equivalence of tensor categories (2.23) after twisting both sides by F^I_ϑ.

The nonsplit case follows via étale descent. □

Remark 3.7.13. As an addendum to the proof of Theorem 2.4.4, we note that (2.23) is compatible with constant term functors, *i.e.* the square below commutes:

$$\begin{array}{ccc} {}^{+}\mathrm{Sat}^I_{G,\mathscr{G}^I} & \xrightarrow{(2.23)} & \mathrm{Rep}^I_{H^I,F^I_\vartheta} \\ \Big\downarrow{\scriptstyle \mathrm{CT}^I_B(\check{\rho})[2\check{\rho}]} & & \Big\downarrow{\scriptstyle \mathrm{res}_{T^I_H}} \\ \mathrm{Sat}^I_{T,\mathscr{G}^I_T} & \xrightarrow{(2.23)} & \mathrm{Rep}^I_{T^I_H,F^I_\vartheta} \end{array}$$

where the right vertical arrow is the restriction along the maximal torus $T^I_H \subset H^I$. (Recall that $\mathrm{CT}^I_B(\check{\rho})[2\check{\rho}]$ is independent of the choice of B according to Remark 3.4.6.)

The compatibility statements in §2.4.5 follow directly from the construction of the equivalence.

4. Global function fields

This section contains our results particular to smooth curves over a finite field.

In §4.1, we propagate the A-gerbe $\mathscr{G}^{\mathrm{I}}$ defined in §2.2 to various moduli spaces associated to a global curve. The crucial observation is that $\mathscr{G}^{\mathrm{I}}$ is canonically trivialized over the moduli stack of Shtukas, allowing us to obtain the space of genuine automorphic forms from its cohomology. This geometric origin of genuine automorphic forms is already indicated by V. Lafforgue in [Laf18, §14] in a narrower context.

In §4.2, we prove a lemma concerning Artin reciprocity. It implies that in the function field context, Weissman's meta-Galois group ([Wei18, §4]) is the central extension associated to the $\{\pm 1\}$-gerbe of theta characteristics. This is a slightly surprising fact, but it follows from very natural considerations.

Finally, we explain in §4.3-4.4 how to extend the arguments of V. Lafforgue [Laf18] (using improvements by Xue [Xue20a], [Xue20b]) to obtain the spectral decomposition of genuine cusp forms defined on covering groups.

4.0.1. Let k be a finite field of cardinality q. For any k-scheme S, we write Fr_{S} for the qth power Frobenius endomorphism of S.

Let X be a smooth, proper, and geometrically connected curve over k. Denote by F its field of fractions and $\mathbb{A}_{\mathrm{F}}$ (resp. $\mathbb{O}_{\mathrm{F}}$) its ring of (resp. integral) adèles.

The coefficient field E is as in §2.0.1. We assume that q has a square root in E which will be fixed: this corresponds to the choice of $\underline{\mathrm{E}}(\frac{1}{2})$ used in the geometric Satake equivalence (see §2.4).

Let $\mathrm{D} \subset \mathrm{X}$ be a k-finite closed subscheme and write $\mathring{\mathrm{X}} := \mathrm{X} \backslash \mathrm{D}$ for its open complement.

Let $\mathrm{G} \to \mathrm{X}$ be a smooth affine group scheme with connected geometric fibers, equipped with an étale metaplectic cover μ defined over $\mathring{\mathrm{X}}$.

4.1. The global A-gerbe.

4.1.1. Let $\mathrm{Bun}_{\mathrm{G,D}}$ denote the stack whose S-points, for any affine k-scheme S, consist of pairs (P, ϕ) where P is a G-torsor over $\mathrm{S} \times \mathrm{X}$ and ϕ is a rigidification of P along $\mathrm{S} \times \mathrm{D}$.

For each nonempty finite set I, let $\mathrm{Hec}^{\mathrm{I}}_{\mathrm{G,D}}$ denote the stack whose S-points consist of an S-point x^{I} of $\mathring{\mathrm{X}}^{\mathrm{I}}$, pairs (P_0, ϕ_0), (P_1, ϕ_1) of G-torsors over $\mathrm{S} \times \mathrm{X}$ rigidified along D, and an isomorphism of them off the union of graphs $\Gamma_{x^{\mathrm{I}}} \subset \mathrm{S} \times \mathrm{X}$.

We refer to such an isomorphisms as a "modification" at x^{I} and denote it by:

$$(\mathrm{P}_0, \phi_0) \overset{x^{\mathrm{I}}}{\sim} (\mathrm{P}_1, \phi_1).$$

4.1.2. If I is equipped with an ordered partition into nonempty finite sets $\mathrm{I} \cong \mathrm{I}_1 \sqcup \cdots \sqcup \mathrm{I}_k$, we write $\widetilde{\mathrm{Hec}}^{\mathrm{I}_1, \cdots, \mathrm{I}_k}_{\mathrm{G,D}}$ for the stack whose S-points consist of an S-point x^{I} of $\mathring{\mathrm{X}}^{\mathrm{I}}$ and modifications:

$$(\mathrm{P}_0, \phi_0) \overset{x^{\mathrm{I}_1}}{\sim} (\mathrm{P}_1, \phi_1) \overset{x^{\mathrm{I}_2}}{\sim} \cdots \overset{x^{\mathrm{I}_k}}{\sim} (\mathrm{P}_k, \phi_k), \tag{4.1}$$

where each x^{I_a} denotes the corresponding S-point of $\mathring{\mathrm{X}}^{\mathrm{I}_a}$ (for $1 \le a \le k$).

Restricting the data (4.1) to the formal disk $\mathrm{D}_{x^{\mathrm{I}}}$ defines a morphism from $\widetilde{\mathrm{Hec}}^{\mathrm{I}_1, \cdots, \mathrm{I}_k}_{\mathrm{G,D}}$ to the local iterated Hecke stack $\widetilde{\mathrm{Hec}}^{\mathrm{I}_1, \cdots, \mathrm{I}_k}_{\mathrm{G}}$ of §2.1. For each $0 \le a \le k$, remembering (P_a, ϕ_a) defines a morphism $p_a : \widetilde{\mathrm{Hec}}^{\mathrm{I}_1, \cdots, \mathrm{I}_k}_{\mathrm{G,D}} \to \mathrm{Bun}_{\mathrm{G,D}}$.

Finally, an S-point of the moduli stack of iterated Shtukas $\mathrm{Sht}^{\mathrm{I}_1, \cdots, \mathrm{I}_k}_{\mathrm{G,D}}$ consists of an S-point (4.1) of $\widetilde{\mathrm{Hec}}^{\mathrm{I}_1, \cdots, \mathrm{I}_k}_{\mathrm{G,D}}$ together with an isomorphism:

$$(\mathrm{P}_k, \phi_k) \cong {}^{\tau}(\mathrm{P}_0, \phi_0) := (\mathrm{Fr}_{\mathrm{S}} \times \mathrm{id}_{\mathrm{X}})^*(\mathrm{P}_0, \phi_0). \tag{4.2}$$

Some of the relevant morphisms are recorded in the diagram below, where the square is Cartesian by definition:

$$\begin{array}{ccc} \mathrm{Sht}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}_{\mathrm{G,D}} & \xrightarrow{p_0} & \mathrm{Bun}_{\mathrm{G,D}} \\ \downarrow & & \downarrow (\mathrm{id},\mathrm{Fr}_{\mathrm{Bun}_{\mathrm{G,D}}}) \\ \widetilde{\mathrm{Hec}}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}_{\mathrm{G,D}} & \xrightarrow{(p_0,p_k)} & \mathrm{Bun}_{\mathrm{G,D}} \times \mathrm{Bun}_{\mathrm{G,D}} \\ \downarrow \mathrm{res} & & \\ \widetilde{\mathrm{Hec}}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}_{\mathrm{G}} & & \\ \downarrow & & \\ \mathring{\mathrm{X}}^{\mathrm{I}} & & \end{array} \tag{4.3}$$

4.1.3. We shall functorially assign an étale A-gerbe $\mathscr{G}_{\mathrm{D}}$ over $\mathrm{Bun}_{\mathrm{G,D}}$ to μ.

Construction. The projection map $p : \mathrm{Bun}_{\mathrm{G,D}} \times \mathrm{X} \to \mathrm{Bun}_{\mathrm{G,D}}$, being proper and smooth of relative dimension one, defines a morphism of complexes $p_*(\underline{\mathrm{A}}(1)[4]) \to \underline{\mathrm{A}}[2]$. Its global section over $\mathrm{Bun}_{\mathrm{G,D}}$ yields the "transgression" map:

$$[\mathrm{X}] : \Gamma(\mathrm{Bun}_{\mathrm{G,D}} \times \mathrm{X}, \underline{\mathrm{A}}(1)[4]) \to \Gamma(\mathrm{Bun}_{\mathrm{G,D}}, \underline{\mathrm{A}}[2]).$$

Let us view the universal G-torsor as a morphism of X-stacks $\mathrm{P} : \mathrm{Bun}_{\mathrm{G,D}} \times \mathrm{X} \to \mathrm{B}_{\mathrm{X}}(\mathrm{G})$ whose base change P_{D} along $\mathrm{D} \subset \mathrm{X}$ is rigidified.

Consider the commutative diagram below:

$$\begin{array}{ccccc} \Gamma_e(\mathrm{B}_{\mathrm{D}}\mathrm{G}, \underline{\mathrm{A}}[2]) & \longrightarrow & \Gamma_e(\mathrm{B}_{\mathrm{X}}\mathrm{G}, \underline{\mathrm{A}}(1)[4]) & \longrightarrow & \Gamma_e(\mathrm{B}_{\mathring{\mathrm{X}}}\mathrm{G}, \underline{\mathrm{A}}(1)[4]) \\ \downarrow (\mathrm{P}_{\mathrm{D}})^* & & \downarrow \mathrm{P}^* & & \\ \Gamma(\mathrm{Bun}_{\mathrm{G,D}} \times \mathrm{D}, \underline{\mathrm{A}}[2]) & \to & \Gamma(\mathrm{Bun}_{\mathrm{G,D}} \times \mathrm{X}, \underline{\mathrm{A}}(1)[4]) & & \\ & & \downarrow [\mathrm{X}] & & \\ & & \Gamma(\mathrm{Bun}_{\mathrm{G,D}}, \underline{\mathrm{A}}[2]) & & \end{array} \tag{4.4}$$

where Γ_e denotes the complex of rigidified sections, and the top row is the triangle induced from the Cousin triangles associated to $\mathrm{D} \to \mathrm{X}$ and $\mathrm{B}_{\mathrm{D}}(\mathrm{G}) \to \mathrm{B}_{\mathrm{X}}(\mathrm{G})$.

The rigidification of P_{D} induces a trivialization of the restriction of $[\mathrm{X}] \cdot \mathrm{P}^*$ in (4.4) to $\Gamma_e(\mathrm{B}_{\mathrm{D}}(\mathrm{G}), \underline{\mathrm{A}}[2])$. Hence $[\mathrm{X}] \cdot \mathrm{P}^*$ factors through a morphism:

$$\Gamma_e(\mathrm{B}_{\mathring{\mathrm{X}}}\mathrm{G}, \underline{\mathrm{A}}(1)[4]) \to \Gamma(\mathrm{Bun}_{\mathrm{G,D}}, \underline{\mathrm{A}}[2]). \tag{4.5}$$

The desired functor $\mu \mapsto \mathscr{G}_\mu$ is obtained from (4.5) upon taking connective truncations and passing to the underlying ∞-groupoids. □

Remark 4.1.4. If $\mathrm{D} = \varnothing$, then the rigidification of μ along $e : \mathring{\mathrm{X}} \to \mathrm{B}_{\mathring{\mathrm{X}}}(\mathrm{G})$ is not needed for the construction of $\mathscr{G}_{\mathrm{D}}$.

Remark 4.1.5. Suppose that G is split reductive.

Inspecting the top row in (4.4) and using the computation of étale cohomology of $\mathrm{B}(\mathrm{G})$ in degrees ≤ 3 (see [Zha22, §5.1]), we see that μ may not extend across D, and when μ extends across some point $x \in \mathrm{D}$, the choice of possible extensions is not unique.

In particular, the étale metaplectic cover μ generally contains more *data* than its restriction to the generic point $\eta \in \mathrm{X}$.

4.1.6. Denote by $\mathrm{K_D}$ the kernel of the projection $\mathrm{G}(\mathbb{O}_\mathrm{F}) \to \mathrm{G}(\mathscr{O}_\mathrm{D})$. The gluing maps yield an inclusion of groupoids:

$$\mathrm{K_D}\backslash \mathrm{G}(\mathbb{A}_\mathrm{F})/\mathrm{G}(\mathrm{F}) \subset \mathrm{Bun}_{\mathrm{G,D}}(k), \tag{4.6}$$

whose essential image consists of pairs (P, ϕ) where P is generically trivial. (This uses the vanishing of $\mathrm{H}^1(\mathscr{O}_x, \mathrm{G})$ for a closed point $x \in \mathrm{X}$, which follows from Lang's isogeny.)

The additional pieces of $\mathrm{Bun}_{\mathrm{G,D}}(k)$ are labeled by the *Shafarevich set*:

$$Ш^1(\mathrm{F}, \mathrm{G}) := \mathrm{Ker}(\mathrm{H}^1(\mathrm{F}, \mathrm{G}) \to \prod_{x\in \mathrm{X}} \mathrm{H}^1(\mathrm{F}_x, \mathrm{G})).$$

Namely, restriction of a G-torsor to the generic point defines a surjective map of pointed groupoids $\mathrm{Bun}_{\mathrm{G,D}}(k) \to Ш^1(\mathrm{F}, \mathrm{G})$ and (4.6) concides with its kernel.

4.1.7. Recall that μ defines a central extension $\widetilde{\mathrm{G}}_\mathrm{F}$ of $\mathrm{G}(\mathbb{A}_\mathrm{F})$ by A, equipped with canonical splittings over $\mathrm{G}(\mathrm{F})$ and $\mathrm{K_D}$ (see §1.5.4 and §2.2.9).

On the other hand, $\mathrm{Tr}(\mathrm{Fr} \mid \mathscr{G}_\mathrm{D})(k)$ is a set-theoretic A-torsor $\widetilde{\mathrm{Bun}}_{\mathrm{G,D}}$ over $\mathrm{Bun}_{\mathrm{G,D}}(k)$. Its restriction along (4.6) is identified with the set-theoretic A-torsor $\mathrm{K_D}\backslash \widetilde{\mathrm{G}}_\mathrm{F}/\mathrm{G}(\mathrm{F})$.

To explain this identification, we note that for each closed point $x \in \mathrm{X}$ with residue field $k_1 \supset k$, the gluing map $\mathrm{G}(\mathrm{F}_x) \to \mathrm{Bun}_{\mathrm{G,D}}(k)$ arises as the k-points of a map:

$$\mathrm{res}(\mathrm{L}(\mathrm{G})_x) \to \mathrm{Bun}_{\mathrm{G,D}}, \tag{4.7}$$

where $\mathrm{res}(\mathrm{L}(\mathrm{G})_x)$ is the Weil restriction of $\mathrm{L}(\mathrm{G})_x$ along $k_1 \supset k$. Indeed, an S-point of $\mathrm{res}(\mathrm{L}(\mathrm{G})_x)$ is equivalent to a section of G over the punctured formal disk around $\mathrm{S}\times x \subset \mathrm{S}\times \mathrm{X}$ (because the map $(\mathrm{S}\times \mathrm{X})_{k_1} \to \mathrm{S}\times \mathrm{X}$ is étale.) This section may be used to glue the trivial bundles on $(\mathrm{S}\times \mathrm{X})\backslash(\mathrm{S}\times x)$ and the formal disk around $\mathrm{S}\times x$ using the Beauville-Laszlo Theorem, defining (4.7).

Comparing the constructions of §2.2.2 and §4.1.3, we see that $\mathscr{G}_\mathrm{D}$ pulls back to $\mathrm{Nm}(\mathscr{G}_x)$ along (4.7), using the notations of Remark 2.2.10. The same remark implies that $\widetilde{\mathrm{Bun}}_{\mathrm{G,D}}(k)$ pulls back to $\widetilde{\mathrm{G}}_x$ along the gluing map $\mathrm{G}(\mathrm{F}_x) \to \mathrm{Bun}_{\mathrm{G,D}}(k)$. If $x \in \mathring{\mathrm{X}}$, this identification is compatible with the sections over $\mathrm{G}(\mathscr{O}_x)$, and if $x \in \mathrm{D}$, it is compatible with the sections over the first congruence subgroup $\mathrm{K}_x := \ker(\mathrm{G}(\mathscr{O}_x) \to \mathrm{G}(k_1))$.

To see that $\widetilde{\mathrm{Bun}}_{\mathrm{G,D}}(k)$ pulls back to $\mathrm{K_D}\backslash \widetilde{\mathrm{G}}_\mathrm{F}/\mathrm{G}(\mathrm{F})$ along (4.6), we perform the same construction for a finite collection of closed points $\{x_i\}$ $(i \in \mathrm{I})$, compare the sections of $\prod_{i\in \mathrm{I}} \widetilde{\mathrm{G}}_{x_i}$ over the subgroup $\mathrm{G}(\mathrm{X}\backslash \bigcup_{i\in\mathrm{I}} x_i) \subset \prod_{i\in\mathrm{I}} \mathrm{G}(\mathrm{F}_{x_i})$, and pass to the colimit as in the definition of $\widetilde{\mathrm{G}}_\mathrm{F}$ [Zha22, §2.2].

4.1.8. Let I be a nonempty finite set equipped with an ordered partition into nonempty finite subsets $\mathrm{I} \cong \mathrm{I}_1 \sqcup \cdots \sqcup \mathrm{I}_k$.

Recall that μ defines an étale A-gerbe $\mathscr{G}^\mathrm{I}$ over $\mathrm{Hec}^\mathrm{I}_\mathrm{G}$. We may form its pullback $\mathscr{G}^{\mathrm{I}_1,\cdots,\mathrm{I}_k} := m^*(\mathscr{G}^\mathrm{I})$ along the composition map $m : \widetilde{\mathrm{Hec}}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}_\mathrm{G} \to \mathrm{Hec}^\mathrm{I}_\mathrm{G}$ (see §2.1.5). Its further pullback along the restriction map in (4.3) defines an A-gerbe $\mathscr{G}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}_\mathrm{D}$ over $\widetilde{\mathrm{Hec}}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}_{\mathrm{G,D}}$.

Let us construct an isomorphism of A-gerbes over $\widetilde{\mathrm{Hec}}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}_{\mathrm{G,D}}$:

$$p_0^*(\mathscr{G}_\mathrm{D}) \otimes p_k^*(\mathscr{G}_\mathrm{D})^{\otimes -1} \cong \mathscr{G}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}_\mathrm{D}. \tag{4.8}$$

Construction. In view of the isomorphism (2.14), it suffices to construct (4.8) in the special case $k = 1$, as the general case will be a product of the isomorphisms associated to each I_a (for $1 \le a \le k$).

Consider now an S-point of $\mathrm{Hec}^\mathrm{I}_{\mathrm{G,D}}$ given by the modification datum $(\mathrm{P}_0, \phi_0) \overset{x^\mathrm{I}}{\sim} (\mathrm{P}_1, \phi_1)$.

Let $i : \Gamma_{x^{\mathrm{I}}} \subset \mathrm{S} \times \mathrm{X}$ and $\hat{i} : \Gamma_{x^{\mathrm{I}}} \subset \mathrm{D}_{x^{\mathrm{I}}}$ denote the closed immersions. The construction of §2.2.2 involves a morphism of complexes:

$$\Gamma(\Gamma_{x^{\mathrm{I}}}, i^!\underline{\mathrm{A}}(1)[4]) \to \Gamma(\mathrm{S}, \underline{\mathrm{A}}[2]). \tag{4.9}$$

The restriction of $p_0^*(\mathscr{G}_{\mathrm{D}}) \otimes p_1^*(\mathscr{G}_{\mathrm{D}})^{\otimes -1}$ to S is defined by the image of the section $\mathrm{P}_0^*(\mu) - \mathrm{P}_1^*(\mu)$ under (4.9), where each P_0, P_1 is viewed as a morphism $\mathrm{S} \times \mathrm{X} \to \mathrm{B}(\mathrm{G})$. It defines a section of $i^!\mathrm{A}(1)[4]$ using the isomorphism $\mathrm{P}_0 \cong \mathrm{P}_1$ off $\Gamma_{x^{\mathrm{I}}}$.

Under the identification $i^!\underline{\mathrm{A}}(1)[4] \cong \hat{i}^!\underline{\mathrm{A}}(1)[4]$, this section is also defined by the restrictions of P_0, P_1 to $\mathrm{D}_{x^{\mathrm{I}}}$ and their identification over $\mathring{\mathrm{D}}_{x^{\mathrm{I}}}$. The image of this section under (4.9) is precisely the restriction of $\mathscr{G}_{\mathrm{D}}^{\mathrm{I}}$ to S. □

4.1.9. Let $\mathrm{I} \cong \mathrm{I}_1 \sqcup \cdots \sqcup \mathrm{I}_k$ be as above.

We shall trivialize the restriction of the A-gerbe $\mathscr{G}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}$ to $\mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}$.

Construction. Indeed, the isomorphism (4.8) exhibits this restriction as the pullback of $\mathscr{G}_{\mathrm{D}} \otimes \mathrm{Fr}^*_{\mathrm{Bun}_{\mathrm{G},\mathrm{D}}}(\mathscr{G}_{\mathrm{D}})^{\otimes -1}$ along the morphism $p_0 : \mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{\mathrm{I}_1,\cdots,\mathrm{I}_k} \to \mathrm{Bun}_{\mathrm{G},\mathrm{D}}$, but there is an isomorphism $\mathrm{Fr}^*_{\mathrm{Bun}_{\mathrm{G},\mathrm{D}}}(\mathscr{G}_{\mathrm{D}}) \cong \mathscr{G}_{\mathrm{D}}$ supplied by (1.14). □

4.1.10. In particular, direct image with compact support along the projection $\mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{\mathrm{I}_1,\cdots,\mathrm{I}_k} \to \mathring{\mathrm{X}}^{\mathrm{I}}$ defines a functor of ∞-categories:

$$\mathrm{Shv}_{\mathscr{G}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}}(\widetilde{\mathrm{Hec}}_{\mathrm{G}}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}) \to \mathrm{Ind}(\mathrm{Shv}(\mathring{\mathrm{X}}^{\mathrm{I}})), \quad \mathscr{F} \mapsto \Gamma_c(\mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}, \mathscr{F}). \tag{4.10}$$

Here, the functor of compactly supported cohomology of a constructible E-sheaf is well-defined because $\mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}$ is a union of quasi-compact open substacks which are ind-algebraic stacks of ind-finite type [Laf18, Lemme 12.19].

4.1.11. For a nonempty finite set I, consider the unit $e_!(\underline{\mathrm{E}})$ of the Satake category defined using the trivialization of $\mathscr{G}^{\mathrm{I}}$ along the unit section of $\mathrm{Hec}_{\mathrm{G}}^{\mathrm{I}}$ (see §2.3).

Recall also the E-vector space $\mathrm{Fun}_c(\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}}, \mathrm{A} \subset \mathrm{E}^\times)$ of genuine functions of compact support on $\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}}$.

We shall construct a canonical isomorphism of ind-constructible sheaves over $\mathring{\mathrm{X}}^{\mathrm{I}}$:

$$\Gamma_c(\mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{\mathrm{I}}, e_!(\underline{\mathrm{E}})) \cong \mathrm{Fun}_c(\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}}, \mathrm{A} \subset \mathrm{E}^\times) \otimes \underline{\mathrm{E}}. \tag{4.11}$$

Construction. The base change of $\mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{\mathrm{I}}$ along the unit section $\mathring{\mathrm{X}}^{\mathrm{I}} \to \mathrm{Hec}_{\mathrm{G}}^{\mathrm{I}}$ is identified with the following fiber product:

$$\begin{array}{ccc} (\mathrm{Bun}_{\mathrm{G},\mathrm{D}})^{\mathrm{Fr}} \times \mathring{\mathrm{X}}^{\mathrm{I}} & \longrightarrow & \mathrm{Bun}_{\mathrm{G},\mathrm{D}} \\ \downarrow & & \downarrow {\scriptstyle (\mathrm{id}, \mathrm{Fr}_{\mathrm{Bun}_{\mathrm{G},\mathrm{D}}})} \\ \mathrm{Bun}_{\mathrm{G},\mathrm{D}} \times \mathring{\mathrm{X}}^{\mathrm{I}} & \xrightarrow{\Delta} & \mathrm{Bun}_{\mathrm{G},\mathrm{D}} \times \mathrm{Bun}_{\mathrm{G},\mathrm{D}} \end{array}$$

We shall obtain (4.11) by playing with two distinct trivializations of the restriction of $\mathscr{G}^{\mathrm{I}}$ to $(\mathrm{Bun}_{\mathrm{G},\mathrm{D}})^{\mathrm{Fr}} \times \mathring{\mathrm{X}}^{\mathrm{I}}$, coming from §4.1.9 respectively the unit section of $\mathrm{Hec}_{\mathrm{G}}^{\mathrm{I}}$.

To wit, the image of $e_!(\underline{\mathrm{E}})$ under (4.10) is calculated as follows: we start with the constant sheaf $\underline{\mathrm{E}}$ over $\mathrm{Bun}_{\mathrm{G},\mathrm{D}} \times \mathring{\mathrm{X}}^{\mathrm{I}}$, view it as twisted by the trivial A-gerbe $\Delta^*(\mathscr{G}_{\mathrm{D}} \boxtimes \mathscr{G}_{\mathrm{D}}^{\otimes -1})$, pull it back to $(\mathrm{Bun}_{\mathrm{G},\mathrm{D}})^{\mathrm{Fr}} \times \mathring{\mathrm{X}}^{\mathrm{I}}$ and view it as twisted by the equivalent A-gerbe $(\mathrm{id}, \mathrm{Fr}_{\mathrm{Bun}_{\mathrm{G},\mathrm{D}}})^*(\mathscr{G}_{\mathrm{D}} \boxtimes \mathscr{G}_{\mathrm{D}}^{\otimes -1})$ but *trivialized by* (1.14), and finally take its !-direct image towards $\mathring{\mathrm{X}}^{\mathrm{I}}$.

The fact that this procedure yields $\mathrm{Fun}_c(\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}}, \mathrm{A} \subset \mathrm{E}^\times) \otimes \underline{\mathrm{E}}$ is observed in §1.4.10. □

4.2. A lemma for $\mathbb{G}_m$.

4.2.1. We assume $\mathrm{D} \neq \varnothing$ in this subsection.

Let $\infty\mathrm{D}$ denote the formal completion of X along D. Write $\eta = \mathrm{Spec}(\mathrm{F})$ for the generic point of X and choose an algebraic closure $\mathrm{F} \subset \bar{\mathrm{F}}$, with $\bar{\eta} \coloneqq \mathrm{Spec}(\bar{\mathrm{F}})$.

An S-point of the stack $\mathrm{Bun}_{\mathrm{G},\infty\mathrm{D}}$ consists of a G-torsor over $\mathrm{S} \times \mathrm{X}$ equipped with a trivialization over the formal disk around $\mathrm{S} \times \mathrm{D} \subset \mathrm{S} \times \mathrm{X}$, or equivalently a G-torsor over $\mathrm{S} \times \mathring{\mathrm{X}}$ equipped with a trivialization over the punctured formal disk around $\mathrm{S} \times \mathrm{D} \subset \mathrm{S} \times \mathrm{X}$. In particular, it is well-defined even when G is only a smooth affine group scheme over $\mathring{\mathrm{X}}$.

4.2.2. The Artin reciprocity map is an isomorphism of topological abelian groups:

$$\mathrm{Art} : \pi_1(\mathring{\mathrm{X}}, \bar{\eta})^{\mathrm{ab}} \cong \mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}}(k)^{\mathrm{profin}}, \tag{4.12}$$

where the target denotes the profinite completion of $\mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}}(k)$. It is normalized so that the geometric Frobenius element $\varphi_x \in \mathrm{Gal}(\bar{k}_x/k_x)$, for each closed point $x \in \mathring{\mathrm{X}}$ with residue field k_x, maps to $\mathscr{O}(x)$. (Note that $\mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}}$ is a scheme when $\mathrm{D} \neq \varnothing$.)

On the other hand, we have the Abel–Jacobi morphism:

$$\mathrm{AJ} : \mathring{\mathrm{X}} \to \mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}}, \quad x \mapsto \mathscr{O}(x),$$

where $\mathscr{O}(x)$ is equipped with its canonical trivialization over $\infty\mathrm{D}$, as $x \notin \mathrm{D}$.

4.2.3. Recall the notion of the trace of Frobenius of an A-gerbe from §1.4.

The following lemma shows that the Abel–Jacobi morphism geometrizes Artin reciprocity on the level of "character A-gerbes".

Lemma 4.2.4. *The following diagram is canonically commutative:*

$$\begin{array}{ccc} \mathrm{Maps}_{\mathbb{Z}}(\mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}}, \mathrm{B}^2(\underline{\mathrm{A}})) & \xrightarrow{\mathrm{AJ}^*} & \mathrm{Maps}(\mathring{\mathrm{X}}, \mathrm{B}^2(\underline{\mathrm{A}})) \\ \downarrow{\scriptstyle \mathrm{Tr}(\mathrm{Fr}|-)(k)} & & \downarrow{\scriptstyle (1.7)} \\ \mathrm{Maps}_{\mathbb{Z}}(\mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}}(k), \mathrm{B}(\mathrm{A})) & \xrightarrow{\mathrm{Art}^*} & \mathrm{CExt}(\pi_1(\mathring{\mathrm{X}}, \bar{\eta}), \mathrm{A}) \end{array} \tag{4.13}$$

(Since A *is finite, every multiplicative* A*-torsor over* $\mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}}(k)$ *descends along its profinite completion, so* Art^* *is well-defined.)*

Proof. We divide the proof into two claims.

Claim 1: (4.13) *is commutative over the neutral component.*

To prove this assertion, it suffices to consider the loop spaces of (4.13) and show that the resulting diagram is commutative.

Namely, given a $\mathbb{Z}$-linear morphism $\mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}} \to \mathrm{B}(\underline{\mathrm{A}})$, or equivalently a commutative multiplicative A-torsor t over $\mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}}$, we need to compare the character $\pi_1(\mathring{\mathrm{X}}, \bar{\eta}) \to \mathrm{A}$ associated to $\mathrm{AJ}^*(t)$ with the pullback of $\mathrm{Tr}(\mathrm{Fr} \mid t)(k)$ along (4.12).

Their equality is a familiar fact in geometric class field theory and follows immediately from the Chebotarev density theorem.

Claim 2: any $\mathbb{Z}$*-linear morphism* $\mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}} \to \mathrm{B}^2(\underline{\mathrm{A}})$ *is (non-canonically) trivial over a finite extension of* k*.*

Indeed, the commutativity of (4.13) can be verified étale locally over $\mathrm{Spec}(k)$, so it will follow from combining the two claims.

To prove Claim 2, it suffices to establish the vanishing:

$$\underline{\mathrm{Ext}}^2(\mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}}, \underline{\mathrm{A}}) = 0, \tag{4.14}$$

where $\underline{\mathrm{Ext}}$ denotes the internal Ext-group for étale sheaves over $\mathrm{Spec}(k)$.

Replacing k by a finite extension if necessary, the reduced subscheme of D becomes a finite nonempty collection of k-points x^{I} of X. We choose an element $i_0 \in \mathrm{I}$ and fit $\mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}}$ into a system of three short exact sequences:

$$\begin{array}{ccccc}
 & & \prod_{i\in\mathrm{I}}\mathrm{Ker}(\mathrm{L}_+(\mathbb{G}_m)_{x^i}\to\mathbb{G}_m) & & \\
 & & \big\downarrow & & \\
 & & \mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}} & & \mathrm{Pic}^0 \\
 & & \big\downarrow & & \big\downarrow \\
\prod_{i\neq i_0}\mathbb{G}_m & \hookrightarrow & \mathrm{Bun}_{\mathbb{G}_m,x^{\mathrm{I}}} & \twoheadrightarrow & \mathrm{Pic} \\
 & & & & \big\downarrow \\
 & & & & \mathbb{Z}
\end{array}$$

Here, $\mathrm{Pic}\cong\mathrm{Bun}_{\mathbb{G}_m,x^{i_0}}$ is the Picard scheme of X and Pic^0 is its neutral component.

It thus suffices to prove that $\underline{\mathrm{Ext}}^2(\mathrm{M},\underline{\mathrm{A}})=0$, for M a pro-unipotent group scheme, $\mathbb{G}_m$, an abelian variety, and $\underline{\mathbb{Z}}$.

The pro-unipotent case and the case $\mathrm{M}=\underline{\mathbb{Z}}$ are clear. For $\mathrm{M}=\mathbb{G}_m$ or an abelian variety, we may assume that $\underline{\mathrm{A}}=\mu_n$, where n is invertible in k. Morphisms $\mathrm{M}\to\mu_n[2]$ of complexes are equivalent to morphisms $\mathrm{M}_{[n]}\to\mathbb{G}_m[1]$, where $\mathrm{M}_{[n]}\subset\mathrm{M}$ is the subgroup scheme of n-torsion elements. However, $\underline{\mathrm{Ext}}^1(\mathrm{M}_{[n]},\mathbb{G}_m)=0$ because $\mathrm{M}_{[n]}$ is finite (locally) free ([GRR72, VIII, Proposition 3.3.1]). □

4.2.5. Recall that for a sheaf of abelian groups M over $\mathrm{Spec}(k)$, the groupoid of $\mathbb{E}_\infty$-monoidal morphisms $\mathrm{M}\to\mathrm{B}^2(\underline{\mathrm{A}})$ fits into the split fiber sequence (1.23).

For $\mathrm{M}=\mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}}$, the splitting supplies us with a retract:

$$\mathrm{Maps}_{\mathbb{E}_\infty}(\mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}},\mathrm{B}^2(\underline{\mathrm{A}}))\to\mathrm{Maps}_{\mathbb{Z}}(\mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}},\mathrm{B}^2(\underline{\mathrm{A}})). \tag{4.15}$$

Observe that the functors denoted by AJ^* and $\mathrm{Tr}(\mathrm{Fr}\mid -)(k)$ in (4.13) are naturally defined on $\mathrm{Maps}_{\mathbb{E}_\infty}(\mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}},\mathrm{B}^2(\underline{\mathrm{A}}))$, but they both factor through the retract (4.15). Indeed, they both depend only on the underlying $\mathbb{E}_1$-monoidal structure.

Combining this observation, the compatibility in §4.1.7, and Lemma 4.2.4, we find two commutative squares:

$$\begin{array}{ccccc}
\mathrm{Maps}_{\mathbb{E}_\infty}(\mathrm{B}_{\mathring{\mathrm{X}}}\mathbb{G}_m,\mathrm{B}^4_{\mathring{\mathrm{X}}}\underline{\mathrm{A}}(1)) & \xrightarrow{\mu\mapsto\mathscr{G}_{\mathrm{D}}} & \mathrm{Maps}_{\mathbb{E}_\infty}(\mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}},\mathrm{B}^2(\underline{\mathrm{A}})) & \xrightarrow{\mathrm{AJ}^*} & \mathrm{Maps}(\mathring{\mathrm{X}},\mathrm{B}^2(\underline{\mathrm{A}})) \\
\big\downarrow{\scriptstyle \mu\mapsto\widetilde{\mathrm{G}}_{\mathrm{F}}} & & \big\downarrow{\scriptstyle \mathrm{Tr}(\mathrm{Fr}|-)(k)} & & \big\downarrow{\scriptstyle (1.7)} \\
\mathrm{Maps}_{\mathbb{Z}}(\mathrm{F}^\times\backslash\mathbb{A}_{\mathrm{F}}^\times/\mathrm{K}_{\infty\mathrm{D}},\mathrm{B}(\mathrm{A})) & \cong & \mathrm{Maps}_{\mathbb{Z}}(\mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}}(k),\mathrm{B}(\mathrm{A})) & \xrightarrow{\mathrm{Art}^*} & \mathrm{CExt}(\pi_1(\mathring{\mathrm{X}},\bar{\eta}),\mathrm{A})
\end{array} \tag{4.16}$$

4.2.6. For the moment, let us assume that $\mathrm{char}(k)\neq 2$ and $\mathrm{A}=\{\pm1\}\subset\mathrm{E}^\times$. We shall use the diagram (4.16) to relate Weissman's meta-Galois group (see [Wei18, §4]) with the $\{\pm1\}$-gerbe of theta characteristics.

Consider the étale metaplectic cover $\mu:\mathrm{B}_{\mathrm{X}}(\mathbb{G}_m)\to\mathrm{B}^4_{\mathrm{X}}(\{\pm1\}^{\otimes2})$ defined by the cocycle:

$$\Lambda\otimes\Lambda\to\mathbb{Z}/2,\quad (1,1)\mapsto 1, \tag{4.17}$$

where $\Lambda\cong\mathbb{Z}$ is the cocharacter lattice of $\mathbb{G}_m$ (see [Zha22, §4.4]).

Its induced topological cover of $\mathbb{A}_{\mathrm{F}}^\times$ is the central extension defined by the cocycle:

$$\mathbb{A}_{\mathrm{F}}^\times\otimes\mathbb{A}_{\mathrm{F}}^\times\to\{\pm1\},\quad (a,b)\mapsto\prod_{x\in\mathrm{X}}\mathrm{Hilb}_x(a,b),$$

where $\mathrm{Hilb}_x : \mathrm{F}_x \otimes \mathrm{F}_x \to \{\pm 1\}$ denotes the quadratic Hilbert symbol at $x \in \mathrm{X}$, equipped with canonical splittings over $\mathrm{F}^\times$ and $\mathbb{O}_\mathrm{F}^\times$.

Therefore, the image of μ under the lower circuit of (4.16) is the meta-Galois group of $\mathring{\mathrm{X}}$ by construction:

$$1 \to \{\pm 1\} \to \widetilde{\pi}_1(\mathring{\mathrm{X}}, \bar{\eta}) \to \pi_1(\mathring{\mathrm{X}}, \bar{\eta}) \to 1. \tag{4.18}$$

Corollary 4.2.7. *If* $\mathrm{char}(k) \neq 2$*, then* (4.18) *is canonically identified with the central extension associated to the* $\{\pm 1\}$*-gerbe* $\omega_{\mathring{\mathrm{X}}}^{1/2}$ *under* (1.7).

Proof. Let us trace the image of μ under the upper circuit of (4.16).

Indeed, μ induces the A-gerbe $\mathscr{G}_\mathrm{D}$ over $\mathrm{Bun}_{\mathbb{G}_m, \infty\mathrm{D}}$ and it suffices to make the identification $\mathrm{AJ}^*(\mathscr{G}_\mathrm{D}) \cong \omega_{\mathring{\mathrm{X}}}^{1/2}$. This isomorphism is supplied by Lemma 3.1.7 for $\lambda = 1 \in \mathbb{Z}$. □

Remark 4.2.8. It follows from Corollary 4.2.7 that the meta-Galois group (4.18) for function fields (global, local, and local integral) is *non-canonically* split, and is functorial with respect to finite separable extensions.

These facts are established by Weissman by different means, see [Wei18, §4.2, 4.4].

The stipulation that, for F of equal characteristic 2, the meta-Galois group is the split extension [Wei18, §4.1] appears to align with the classical fact that a canonical theta characteristic exists over X when $\mathrm{char}(k) = 2$.

4.2.9. Let T be a split torus and $\mathrm{T}^\sharp \to \mathrm{T}$ be the isogeny defined in §3.1.1.

Since the restriction of μ to $\mathrm{B}_{\mathring{\mathrm{X}}}(\mathrm{T}^\sharp)$ acquires an $\mathbb{E}_\infty$-monoidal structure [Zha22, §4.6], the restriction of $\mathscr{G}_\mathrm{D}$ to $\mathrm{Bun}_{\mathrm{T}^\sharp, \mathrm{D}}$ has the structure of an $\mathbb{E}_\infty$-monoidal morphism $\mathrm{Bun}_{\mathrm{T}^\sharp, \mathrm{D}} \to \mathrm{B}^2(\underline{\mathrm{A}})$. In particular, its trace of Frobenius defines a $\mathbb{Z}$-linear morphism:

$$\mathrm{Tr}(\mathrm{Fr} \mid \mathscr{G}_\mathrm{D}) : \mathrm{Bun}_{\mathrm{T}^\sharp, \infty\mathrm{D}}(k) \to \mathrm{B}(\mathrm{E}^\times). \tag{4.19}$$

The Artin reciprocity map (4.12) induces an equivalence of groupoids between $\mathbb{Z}$-linear morphisms $\mathrm{Bun}_{\mathrm{T}^\sharp, \infty\mathrm{D}}(k) \to \mathrm{B}(\mathrm{E}^\times)$ and $\mathbb{Z}$-linear morphisms $\mathrm{Weil}(\mathring{\mathrm{X}}, \bar{\eta})^{\mathrm{ab}} \to \mathrm{B}(\mathrm{H}(\mathrm{E}))$, where H is identified with the E-torus dual to $\mathrm{T}^\sharp$. The latter groupoid admits a functor to central extensions of $\mathrm{Weil}(\mathring{\mathrm{X}}, \bar{\eta})$ by $\mathrm{H}(\mathrm{E})$.

Using an argument similar to the proof of Corollary 4.2.7, we identify the image of (4.19) under this functor with (the Weil form of) the L-group ${}^\mathrm{L}\mathrm{H}_{\mathring{\mathrm{X}}, \vartheta}$.

Thus we obtain a canonical bijection:

$$\left\{ \begin{matrix} \text{genuine characters} \\ \widetilde{\mathrm{Bun}}_{\mathrm{T}^\sharp, \infty\mathrm{D}} \to \mathrm{E}^\times \end{matrix} \right\} \cong \left\{ \begin{matrix} \text{sections of} \\ {}^\mathrm{L}\mathrm{H}_{\mathring{\mathrm{X}}, \vartheta} \to \mathrm{Weil}(\mathring{\mathrm{X}}, \bar{\eta}) \end{matrix} \right\}, \tag{4.20}$$

as both sides correspond to null-homotopies of (4.19).

Taking colimit of the bijection (4.20) over increasing D, we obtain a bijection between genuine characters $\mathrm{T}^\sharp(\mathrm{F}) \backslash \widetilde{\mathrm{T}}^\sharp \to \mathrm{E}^\times$ and sections of ${}^\mathrm{L}\mathrm{H}_{\eta, \vartheta} \to \mathrm{Weil}(\eta, \bar{\eta})$. We thus recover a part of the Langlands correspondence for covering groups of split tori ([Wei18, Part 3]).

4.3. Cusp forms.

4.3.1. Suppose that the restriction of G to $\mathring{\mathrm{X}}$ is reductive and its restriction to $\mathrm{Spec}(\mathscr{O}_x)$ for each closed point $x \in \mathrm{D}$ is parahoric.

4.3.2. Let Z denote the radical of $\mathrm{G}_{\mathring{\mathrm{X}}}$, viewed as an affine group scheme (in fact, a torus) over $\mathring{\mathrm{X}}$. The symmetric form b associated to μ restricts to a bilinear form $\Lambda_\mathrm{Z} \otimes \Lambda \to \underline{\mathrm{A}}(-1)$, and we set $\Lambda_\mathrm{Z}^\sharp \subset \Lambda_\mathrm{Z}$ to be its kernel. It corresponds to an isogeny of tori $\mathrm{Z}^\sharp \to \mathrm{Z}$. The group scheme $\mathrm{Z}^\sharp$ plays the role of the "center" in the metaplectic context.

The stack $\mathrm{Bun}_{\mathrm{Z}^\sharp, \infty\mathrm{D}}$ is defined as in §4.2.1.

Remark 4.3.3. The E-torus dual to $Z^\sharp$ is canonically identified with the maximal abelian quotient H^{ab} of H.

Indeed, a character of H^{ab} is by definition a section of $\Lambda^\sharp$ which pairs to zero with $\mathrm{ord}(Q(\alpha))^{-1}\check{\alpha}$ for each $\alpha \in \Delta$. This is precisely a section of $\Lambda^\sharp_Z \cong \Lambda^\sharp \cap \Lambda_Z$.

4.3.4. Recall that the restriction $\mu_{Z^\sharp}$ of μ to $B(Z^\sharp)$ acquires a canonical $\mathbb{E}_\infty$-monoidal structure ([Zha22, §4.6]).

Furthermore, μ is $B(Z^\sharp)$-equivariant against $\mu_{Z^\sharp}$ with respect to the action of $B(Z^\sharp)$ on $B(G)$ ([Zha22, §5.4]).

These observations imply that $\widetilde{\mathrm{Bun}}_{Z^\sharp,\infty D}$ has a commutative multiplicative structure and acts naturally on $\widetilde{\mathrm{Bun}}_{G,D}$.

4.3.5. Let us now choose a subgroup $\Xi \subset \widetilde{\mathrm{Bun}}_{Z^\sharp,D}$ which maps isomorphically onto its image in $\mathrm{Bun}_{Z^\sharp,D}(k)$ and such that $\mathrm{Bun}_{Z^\sharp,D}(k)/\Xi$ is finite. (One may think of Ξ as the kernel of a genuine character on $\widetilde{\mathrm{Bun}}_{Z^\sharp,D}$ with finite image.)

Then Ξ is the k-points of a discrete subscheme to be denoted with the same letter:

$$\Xi \subset \mathrm{Tr}(\mathrm{Fr} \mid \mathscr{G}_{Z^\sharp,D}),$$

which maps isomorphically onto its image in $(\mathrm{Bun}_{Z^\sharp,D})^{\mathrm{Fr}}$.

4.3.6. Consider the E-vector space of compactly supported functions $f : \widetilde{\mathrm{Bun}}_{G,D}/\Xi \to E$ such that $f(x \cdot a) = f(x) \cdot a$ for each $x \in \widetilde{\mathrm{Bun}}_{G,D}/\Xi$ and $a \in A$:

$$\mathrm{Fun}_c(\widetilde{\mathrm{Bun}}_{G,D}/\Xi, A \subset E^\times). \tag{4.21}$$

We shall call such functions *genuine automorphic forms*.

They define genuine automorphic forms in the sense of §1.5.5 as follows: using the compatibility of §4.1.7, we may restrict along (4.6) to obtain compactly supported genuine functions over $K_D\backslash\widetilde{G}_F/G(F)\Xi$.

4.3.7. Suppose that P_η is a parabolic subgroup of the restriction G_η of G to the generic point $\eta \subset X$. Then P_η extends uniquely to a parabolic subgroup P of $G_{\mathring{X}}$. Let $P \twoheadrightarrow M$ denote its Levi quotient.

The restriction of μ to $B_{\mathring{X}}(P)$ canonically descends to a rigidified section of $B^4A(1)$ over $B_{\mathring{X}}(M)$. In particular, the canonical maps $G \leftarrow P \to M$ induce maps of stacks:

$$\widetilde{\mathrm{Bun}}_{G,\infty D} \leftarrow \widetilde{\mathrm{Bun}}_{P,\infty D} \rightarrow \widetilde{\mathrm{Bun}}_{M,\infty D} \tag{4.22}$$

The E-vectors space of *genuine cusp forms*:

$$\mathrm{Fun}_{\mathrm{cusp}}(\widetilde{\mathrm{Bun}}_{G,D}/\Xi, A \subset E^\times) \tag{4.23}$$

is defined to be the subspace of (4.21) consisting of elements which vanish under the integral transform along (4.22) for all proper parabolic subgroups $P_\eta \subset G_\eta$. (The definition of the integral transform requires fixing a Haar measure on the appropriate unipotent groups, but its vanishing is independent of this choice.)

Remark 4.3.8. Over the subspace of (4.21) of functions supported on $K_D\backslash\widetilde{G}(\mathbb{A}_F)/G(F)\Xi$, the cuspidality condition coincides with the one from [BJ79, §3.3].

To check their agreement, one needs the vanishing of $Ш^1(F, N_P)$ for the unipotent radical $N_P \subset P$, which follows from [ABD+66, Exposé XXVI, Corollaire 2.2].

4.3.9. Recall the definition of Hecke operators in §2.4.8: for each closed point $x \in \mathring{\mathrm{X}}$ and $\mathrm{V} \in \mathrm{Rep}^{\mathrm{alg}}({}^{\mathrm{L}}\mathrm{H}_{x,\vartheta})$, we have associated an element:

$$h_{\mathrm{V},x} \in \mathrm{Fun}_c(\mathrm{G}(\mathscr{O}_x)\backslash\widetilde{\mathrm{G}}_x/\mathrm{G}(\mathscr{O}_x), \mathrm{A} \subset \mathrm{E}^{\times}).$$

It acts on the vector space (4.23) via convolution $(h_{\mathrm{V},x}, f) \mapsto h_{\mathrm{V},x} \star f$ along the multiplication map $\widetilde{\mathrm{G}}_x \times \widetilde{\mathrm{G}}_{\mathrm{F}} \to \widetilde{\mathrm{G}}_{\mathrm{F}}$.

4.3.10. We are now ready to state the Langlands parametrization of genuine cusp forms on $\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}}/\Xi$.

The coefficient field E is taken to be $\overline{\mathbb{Q}}_\ell$. We recall that that data (G, μ) over $\mathring{\mathrm{X}}$ define an L-group as a short exact sequence (see (1.28)):

$$1 \to \mathrm{H}_{\bar{\eta}}(\overline{\mathbb{Q}}_\ell) \to {}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}},\vartheta} \to \pi_1(\mathring{\mathrm{X}}, \bar{\eta}) \to 1.$$

Theorem 4.3.11. *Assume* $\mathrm{D} \neq \varnothing$. *There is a canonical decomposition:*

$$\mathrm{Fun}_{\mathrm{cusp}}(\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}}/\Xi, \mathrm{A} \subset \overline{\mathbb{Q}}_\ell^{\times}) \cong \bigoplus_{[\sigma]} \mathbf{H}_{\mathrm{D},[\sigma]}, \tag{4.24}$$

where $[\sigma]$ *ranges over* $\mathrm{H}_{\bar{\eta}}(\overline{\mathbb{Q}}_\ell)$*-conjugacy classes of sections of* ${}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}},\vartheta} \to \pi_1(\mathring{\mathrm{X}}, \bar{\eta})$.

For each $x \in \mathring{\mathrm{X}}$ *and* $\mathrm{V} \in \mathrm{Rep}^{\mathrm{alg}}({}^{\mathrm{L}}\mathrm{H}_{x,\vartheta})$, *the summand* $\mathbf{H}_{\mathrm{D},[\sigma]}$ *is an eigenspace for* $h_{\mathrm{V},x}$ *with eigenvalue* $\mathrm{Tr}([\sigma_x] \cdot \varphi_x \mid \mathrm{V})$, *where* $\varphi_x \in \pi_1(x, \bar{x})$ *denotes the geometric Frobenius and* $[\sigma_x]$ *denotes the* $\mathrm{H}_{\bar{\eta}}(\overline{\mathbb{Q}}_\ell)$*-conjugacy class of sections of* ${}^{\mathrm{L}}\mathrm{H}_{x,\vartheta} \to \pi_1(x, \bar{x})$ *induced from* $[\sigma]$.

4.4. Excursion.

4.4.1. In this subsection, we prove Theorem 4.3.11. The notations are as in §4.3.

In the course of the proof, we will work with a sufficiently large subfield $\mathrm{E} \subset \overline{\mathbb{Q}}_\ell$ finite over $\mathbb{Q}_\ell$ and make our choice $q^{1/2} \in \mathrm{E}^{\times}$.

4.4.2. For a nonempty finite set I equipped with an ordered partition $\mathrm{I} \cong \mathrm{I}_1 \sqcup \cdots \sqcup \mathrm{I}_k$, we specialize the commutative diagram (4.3) to $\mathrm{G} = \mathrm{Z}^{\sharp}$ and restrict along the unit section of the Hecke stacks.

We record this commutative diagram of stacks, along with the A-gerbes defined by $\mu_{\mathrm{Z}^\sharp}$ on some of them:

$$\begin{array}{ccccc}
 & & (\mathrm{Bun}_{\mathrm{Z}^\sharp,\mathrm{D}})^{\mathrm{Fr}} \times \mathring{\mathrm{X}}^{\mathrm{I}} & \xrightarrow{p_0} & \mathrm{Bun}_{\mathrm{Z}^\sharp,\mathrm{D}} & & \\
 & & \downarrow & & \downarrow{\scriptstyle (\mathrm{id},\mathrm{Fr}_{\mathrm{Bun}_{\mathrm{Z}^\sharp,\mathrm{D}}})} & & \\
 & & \mathrm{Bun}_{\mathrm{Z}^\sharp,\mathrm{D}} \times \mathring{\mathrm{X}}^{\mathrm{I}} & \xrightarrow{(p_0,p_k)} & \mathrm{Bun}_{\mathrm{Z}^\sharp,\mathrm{D}} \times \mathrm{Bun}_{\mathrm{Z}^\sharp,\mathrm{D}} & \text{---} & \mathscr{G}_{\mathrm{Z}^\sharp,\mathrm{D}} \boxtimes \mathscr{G}_{\mathrm{Z}^\sharp,\mathrm{D}}^{\otimes -1} \\
 & & \downarrow{\scriptstyle \mathrm{res}} & & & & \\
e^*(\mathscr{G}_{\mathrm{Z}^\sharp}^{\mathrm{I}}) & \text{---} & \mathrm{B}_{\mathring{\mathrm{X}}^{\mathrm{I}}}(\mathrm{L}_+^{\mathrm{I}}(\mathrm{Z}^\sharp)) & & & & \\
 & & \downarrow & & & & \\
 & & \mathring{\mathrm{X}}^{\mathrm{I}} & & & &
\end{array} \tag{4.25}$$

The left column consists of strictly commutative Picard stacks over $\mathring{\mathrm{X}}^{\mathrm{I}}$, and the right column consists of those over k. Morphisms in (4.25) are compatible with these structures, and the A-gerbes admit $\mathbb{E}_\infty$-monoidal structures.

The action of $\mathrm{Z}^\sharp$ on G induces an action of each term in (4.25) on the corresponding term in (4.3), the morphisms among them being equivariant.

4.4.3. Recall that $e^*(\mathscr{G}^{\mathrm{I}}_{\mathrm{Z}^\sharp})$ is canonically trivialized as an A-gerbe over $\mathrm{B}_{\mathring{\mathrm{X}}^{\mathrm{I}}}(\mathrm{L}^{\mathrm{I}}_+(\mathrm{Z}^\sharp))$ equipped with an $\mathbb{E}_\infty$-monoidal structure.

Lemma 4.4.4. *The forgetful functor out of* $\mathrm{B}_{\mathring{\mathrm{X}}^{\mathrm{I}}}(\mathrm{L}^{\mathrm{I}}_+(\mathrm{Z}^\sharp))$*-equivariant objects in* $\widetilde{\mathrm{Sat}}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}$ *is an equivalence of categories:*

$$(\widetilde{\mathrm{Sat}}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}})^{\mathrm{B}_{\mathring{\mathrm{X}}^{\mathrm{I}}}(\mathrm{L}^{\mathrm{I}}_+(\mathrm{Z}^\sharp))} \cong \widetilde{\mathrm{Sat}}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}. \tag{4.26}$$

Proof. The forgetful functor is fully faithful, as $\mathrm{B}_{\mathring{\mathrm{X}}^{\mathrm{I}}}(\mathrm{L}^{\mathrm{I}}_+(\mathrm{Z}^\sharp))$ may be written as an inverse limit of connected smooth algebraic stacks.

To show that it is essentially surjective, it suffices to prove that for each $\mathscr{F} \in \widetilde{\mathrm{Sat}}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}$, there is an isomorphism relating its pullback under the action and projection maps:

$$\mathrm{act}^*(\mathscr{F}) \cong \mathrm{pr}^*(\mathscr{F}), \quad \text{over } \mathrm{B}_{\mathring{\mathrm{X}}^{\mathrm{I}}}(\mathrm{L}^{\mathrm{I}}_+(\mathrm{Z}^\sharp)) \times_{\mathring{\mathrm{X}}^{\mathrm{I}}} \widetilde{\mathrm{Hec}}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}_{\mathrm{G}}, \tag{4.27}$$

extending the natural one over $e \times \widetilde{\mathrm{Hec}}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}_{\mathrm{G}}$, the extension being necessarily unique.

By universal local acyclicity, it is enough to show that such an extension exists over the pairwise disjoint locus of $\mathring{\mathrm{X}}^{\mathrm{I}}$ ([HS21, Theorem 6.8]), and furthermore over any $\bar{k}$-point of the latter. The pointwise statement is a consequence of Proposition 3.3.7. □

4.4.5. Using Lemma 4.4.4, we see that the pullback of any object $\mathscr{F} \in \widetilde{\mathrm{Sat}}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}}$ to $\mathrm{Sht}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}_{\mathrm{G},\mathrm{D}}$ along the vertical map in (4.3) acquires a $(\mathrm{Bun}_{\mathrm{Z}^\sharp,\mathrm{D}})^{\mathrm{Fr}}$-equivariance structure.

In particular, taking direct image with compact support along the projection $\mathrm{Sht}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}_{\mathrm{G},\mathrm{D}}/\Xi \to \mathring{\mathrm{X}}^{\mathrm{I}}$ defines a functor:

$$\widetilde{\mathrm{Sat}}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}_{\mathrm{G},\mathscr{G}^{\mathrm{I}}} \to \mathrm{Ind}(\mathrm{Shv}(\mathring{\mathrm{X}}^{\mathrm{I}})), \quad \mathscr{F} \mapsto \Gamma_c(\mathrm{Sht}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}_{\mathrm{G},\mathrm{D}}/\Xi, \mathscr{F}). \tag{4.28}$$

(Note that $\mathrm{Sht}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}_{\mathrm{G},\mathrm{D}}/\Xi$ is an ind-algebraic stack of ind-finite type.)

The isomorphism (4.11) identifies the image of $e_!(\underline{\mathrm{E}})$ under (4.28) (for $k = 1$) with the constant sheaf over $\mathring{\mathrm{X}}^{\mathrm{I}}$ with values in $\mathrm{Fun}_c(\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}}/\Xi, \mathrm{A} \subset \mathrm{E}^\times)$.

4.4.6. Finally, we summarize the arguments of V. Lafforgue [Laf18] and Xue [Xue20b], which establish the spectral decomposition (4.24), taking as input the Satake functors $\mathscr{S}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}$ (2.32) and the cohomology of Shtukas (4.28). The notion of Shtukas and the method to construct representations of copies of $\pi_1(\mathring{\mathrm{X}}, \bar{\eta})$ using partial Frobenii originated in Drinfeld's work on GL_2 [Dri87b] [Dri88] [Dri87a].

This summary is only included to give detailed references to the results of [Laf18] and [Xue20b]. It contains no originality whatsoever.

Proof of Theorem 4.3.11. For a nonempty finite set I, composing the Satake functor $\mathscr{S}^{\mathrm{I}}$ with (4.28) (for $k = 1$) and taking the middle cohomology group H^0 yields a functor:

$$\mathrm{Rep}^{\mathrm{alg}}(({}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}},\vartheta})^{\mathrm{I}}) \to \mathrm{Ind}(\mathrm{Shv}(\mathring{\mathrm{X}}^{\mathrm{I}})), \quad \mathrm{W} \mapsto \mathscr{H}_{\mathrm{I},\mathrm{W}} := \mathrm{H}^0_c(\mathrm{Sht}^{\mathrm{I}}_{\mathrm{G},\mathrm{D}}/\Xi, \mathscr{S}^{\mathrm{I}}(\mathrm{W})). \tag{4.29}$$

Note that there are fully faithful functors:

$$\mathrm{Ind}(\mathrm{Rep}(\pi_1(\mathring{\mathrm{X}}, \bar{\eta})^{\mathrm{I}})) \subset \mathrm{Ind}(\mathrm{Lis}(\mathring{\mathrm{X}}^{\mathrm{I}})) \subset \mathrm{Ind}(\mathrm{Shv}(\mathring{\mathrm{X}}^{\mathrm{I}})),$$

where $\mathrm{Rep}(\pi_1(\mathring{\mathrm{X}}, \bar{\eta})^{\mathrm{I}})$ denotes the category of finite-dimensional continuous E-linear representations of $\pi_1(\mathring{\mathrm{X}}, \bar{\eta})^{\mathrm{I}}$.

Claim: $\mathscr{H}_{\mathrm{I},\mathrm{W}}$ belongs to $\mathrm{Ind}(\mathrm{Rep}(\pi_1(\mathring{\mathrm{X}}, \bar{\eta})^{\mathrm{I}}))$.

The claim is proved using the action of partial Frobenii on $\mathscr{H}_{\mathrm{I,W}}$. To wit, for an ordered partition $\mathrm{I} \cong \mathrm{I}_1 \sqcup \cdots \sqcup \mathrm{I}_k$ into nonempty finite subsets, we have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Sht}^{\mathrm{I}_1,\mathrm{I}_2,\cdots,\mathrm{I}_k}_{\mathrm{G,D}}/\Xi & \xrightarrow{\mathrm{Fr}_{\mathrm{I}_1}} & \mathrm{Sht}^{\mathrm{I}_2,\cdots,\mathrm{I}_k,\mathrm{I}_1}_{\mathrm{G,D}}/\Xi \\ \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} \\ (\prod_{1\le a\le k} \mathrm{Hec}^{\mathrm{I}_a}_{\mathrm{G}})/\mathrm{B}_{\mathring{\mathrm{X}}^{\mathrm{I}}}(\mathrm{L}^{\mathrm{I}}_+(\mathrm{Z}^\sharp)) & \xrightarrow{\mathrm{Fr}_{\mathrm{I}_1}} & (\prod_{1\le a\le k} \mathrm{Hec}^{\mathrm{I}_1}_{\mathrm{G}})/\mathrm{B}_{\mathring{\mathrm{X}}^{\mathrm{I}}}(\mathrm{L}^{\mathrm{I}}_+(\mathrm{Z}^\sharp)) \\ \downarrow & & \downarrow \\ \mathring{\mathrm{X}}^{\mathrm{I}} & \xrightarrow{\mathrm{Fr}_{\mathrm{I}_1}} & \mathring{\mathrm{X}}^{\mathrm{I}} \end{array} \tag{4.30}$$

where the top horizontal morphism sends the data (4.1), (4.2) to:

$$(\mathrm{P}_1, \phi_1) \overset{x^{\mathrm{I}_2}}{\sim} \cdots \overset{x^{\mathrm{I}_k}}{\sim} {}^\tau(\mathrm{P}_0, \phi_0) \overset{{}^\tau x^{\mathrm{I}_1}}{\sim} {}^\tau(\mathrm{P}_1, \phi_1),$$

and the middle and lower horizontal morphisms are the *partial Frobenii*, acting as the Frobenius on the factor corresponding to I_1 and the identity on the remaining factors.

We view $\mathscr{S}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}$ as valued in the category of *untwisted* perverse sheaves over $\mathrm{Sht}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}_{\mathrm{G,D}}/\Xi$, by pulling back along the restriction maps to the local Hecke stack and using the trivialization of the A-gerbe $\mathscr{G}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}$ over $\mathrm{Sht}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}_{\mathrm{G,D}}$ constructed in §4.1.9. Perversity of the pullback follows from the smoothness and dimension count in [Laf18, §2].

The trivialization of $\mathscr{G}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}$ over $\mathrm{Sht}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}_{\mathrm{G,D}}$ is $\mathrm{Fr}_{\mathrm{I}_1}$-equivariant in the sense that it commutes with the canonical isomorphism $\mathrm{Fr}^*_{\mathrm{I}_1}(\mathscr{G}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}) \cong \mathscr{G}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}$. By the construction of $\mathscr{S}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}$ and the commutative diagram (4.30), we obtain a natural isomorphism:

$$(\mathrm{Fr}_{\mathrm{I}_1})^* \mathscr{S}^{\mathrm{I}_2,\cdots,\mathrm{I}_1}(\mathrm{W}) \cong \mathscr{S}^{\mathrm{I}_1,\cdots,\mathrm{I}_k}(\mathrm{W}). \tag{4.31}$$

Since the outer square of (4.30) is Cartesian up to universal homeomorphisms, (4.31) induces an isomorphism:

$$\mathrm{F}_{\mathrm{I}_1} : (\mathrm{Fr}_{\mathrm{I}_1})^* \mathscr{H}_{\mathrm{I,W}} \cong \mathscr{H}_{\mathrm{I,W}}. \tag{4.32}$$

By re-ordering the partition of I, we obtain similar isomorphisms $\mathrm{F}_{\mathrm{I}_2}, \cdots, \mathrm{F}_{\mathrm{I}_k}$. One sees as in [Laf18, §3-4] that the isomorphisms $\mathrm{F}_{\mathrm{I}_1}, \cdots, \mathrm{F}_{\mathrm{I}_k}$ pairwise commute and their composition equals the canonical identification of $\mathscr{H}_{\mathrm{I,W}}$ with its Frobenius pullback.

Using (4.32), we construct the following endomorphism as in [Laf18, §12.3.3]:

$$\mathrm{S}_{\mathrm{V},x} \in \mathrm{End}(\mathscr{H}_{\mathrm{I,W}}), \quad \text{for } x \in \mathring{\mathrm{X}} \text{ and } \mathrm{V} \in \mathrm{Rep}^{\mathrm{alg}}({}^{\mathrm{L}}\mathrm{H}_{x,\vartheta}).$$

Note that V (unlike W) is a representation of the *local* L-group, so the associated "creation" and "annihilation" operators, corresponding to the unit and counit of V, are only defined over the subscheme $\mathring{\mathrm{X}}^{\mathrm{I}} \times x^{\{1,2\}} \subset \mathring{\mathrm{X}}^{\mathrm{I}\sqcup\{1,2\}}$. (Informally, $\mathrm{S}_{\mathrm{V},x}$ is the trace of the Frobenius endomorphism on "a copy of V inserted at x".)

The fact that $\mathrm{S}_{\mathrm{V},x}$ restricts to the action of the Hecke operator $h_{\mathrm{V},x}$ over $(\mathring{\mathrm{X}}\backslash x)^{\mathrm{I}}$ is proved as in [Laf18, §6].

Furthermore, the E-sheaf $\mathscr{H}_{\mathrm{I}\sqcup\{0\},\mathrm{W}\boxtimes\mathrm{V}}$ is well defined over $\mathring{\mathrm{X}}^{\mathrm{I}} \times x$, and (4.32) induces an endomorphism $(\mathrm{F}_{\{0\}})^{\deg(x)}$ of it. The argument of [Laf18, §7] gives the following identity (to be thought of as a Cayley–Hamilton theorem for $(\mathrm{F}_{\{0\}})^{\deg(x)}$):

$$\sum_{i=0}^{\dim(\mathrm{V})} (-1)^i \mathrm{S}_{\wedge^{\dim(\mathrm{V})-i}(\mathrm{V}),x} \cdot ((\mathrm{F}_{\{0\}})^{\deg(x)})^i = 0. \tag{4.33}$$

The isomorphisms (4.32), together with (4.33), imply the claim by [Xue20b, Proposition 1.3.4, Theorem 4.2.3].

Next, we define the subsheaf of cuspidal cohomology $\mathscr{H}_{\mathrm{I,W,cusp}} \subset \mathscr{H}_{\mathrm{I,W}}$ either by [Laf18, §12.3.4] or by a generalization of [Xue20b, §7] (which is stated for split reductive groups). It belongs to $\mathrm{Rep}(\pi_1(\mathring{\mathrm{X}},\bar{\eta})^{\mathrm{I}})$. Thus (4.29) induces a system of functors:

$$\mathrm{Rep}^{\mathrm{alg}}(({}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}},\vartheta})^{\mathrm{I}}) \to \mathrm{Rep}(\pi_1(\mathring{\mathrm{X}},\bar{\eta})^{\mathrm{I}}), \quad \mathrm{W} \mapsto \mathscr{H}_{\mathrm{I,W,cusp}}, \tag{4.34}$$

indexed by nonempty finite sets I, which are compatible with surjections of such. As noted in §4.4.5, the object $\mathscr{H}_{\mathrm{I},\mathbf{1},\mathrm{cusp}}$ associated to the trivial representation $\mathbf{1}$ is isomorphic to the (finite-dimensional) E-vector space:

$$\mathrm{Fun}_{\mathrm{cusp}}(\widetilde{\mathrm{Bun}}_{\mathrm{G,D}}/\Xi, \mathrm{A} \subset \mathrm{E}^{\times}), \tag{4.35}$$

equipped with the trivial $\pi_1(\mathring{\mathrm{X}},\bar{\eta})^{\mathrm{I}}$-action.

Using the construction of [Laf18, §9-10], the system of functors (4.34) equips the E-vector space (4.35) with the action of a commutative E-algebra $\mathscr{B}$ (of "excursion operators"), and [Laf18, §11] associates to each E-point of $\mathrm{Spec}(\mathscr{B})$ an $\mathrm{H}(\mathrm{E}_1)$-conjugacy class of sections of ${}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}},\vartheta} \to \pi_1(\mathring{\mathrm{X}},\bar{\eta})$ for some finite extension $\mathrm{E} \subset \mathrm{E}_1$ in $\overline{\mathbb{Q}}_\ell$.

Finally, the desired decomposition (4.24) is the decomposition of (4.35) according to its support in $\mathrm{Spec}(\mathscr{B} \otimes_{\mathrm{E}} \overline{\mathbb{Q}}_\ell)$. The action of the Hecke operator $h_{\mathrm{V},x}$ on the summand $\mathbf{H}_{\mathrm{D},[\sigma]}$ is calculated as in [Laf18, §11]. ☐

Remark 4.4.7. It follows from the construction that each conjugacy class $[\sigma]$ appearing in (4.24) is associated to a section $\sigma : \Gamma \to {}^{\mathrm{L}}\mathrm{H}_{\Gamma,\vartheta}$ of the finite form of the L-group (see Remark 1.6.14) and a finite extension $\mathbb{Q}_\ell \subset \mathrm{E}$ in $\overline{\mathbb{Q}}_\ell$:

$$1 \to \mathrm{H}_{\bar{\eta}}(\mathrm{E}) \to {}^{\mathrm{L}}\mathrm{H}_{\Gamma,\vartheta} \to \Gamma \to 1.$$

Furthermore, the Zariski closure of $\sigma(\Gamma) \subset {}^{\mathrm{L}}\mathrm{H}_{\Gamma,\vartheta}$ is a (possibly disconnected) reductive group, when ${}^{\mathrm{L}}\mathrm{H}_{\Gamma,\vartheta}$ is equipped with the algebraic structure induced from $\mathrm{H}_{\bar{\eta}}$.

4.4.8. The spectral decomposition (4.24) is compatible with inclusions of nonempty k-finite closed subscheme $\mathrm{D} \subset \mathrm{D}_1$ of X, *i.e.* the following diagram is commutative:

$$\begin{array}{ccc} \mathbf{H}_{\mathrm{D},[\sigma]} & \subset & \mathrm{Fun}_{\mathrm{cusp}}(\widetilde{\mathrm{Bun}}_{\mathrm{G,D}}/\Xi, \mathrm{A} \subset \overline{\mathbb{Q}}_\ell^{\times}) \\ \downarrow & & \downarrow \\ \mathbf{H}_{\mathrm{D}_1,[\sigma]} & \subset & \mathrm{Fun}_{\mathrm{cusp}}(\widetilde{\mathrm{Bun}}_{\mathrm{G,D}_1}/\Xi, \mathrm{A} \subset \overline{\mathbb{Q}}_\ell^{\times}) \end{array} \tag{4.36}$$

where the left vertical arrow is induced from $\pi_1(\mathring{\mathrm{X}}_1,\bar{\eta}) \to \pi_1(\mathring{\mathrm{X}},\bar{\eta})$, for $\mathring{\mathrm{X}}_1 := \mathrm{X}\backslash\mathrm{D}_1$, and the right vertical arrow is the restriction along $\widetilde{\mathrm{Bun}}_{\mathrm{G,D}_1} \to \widetilde{\mathrm{Bun}}_{\mathrm{G,D}}$.

The generic version of the spectral decomposition (0.3) asserted in Theorem A is a formal consequence of (4.24) and the compatibility (4.36).

Namely, the existence of parahoric models [BT84], combined with [Zha22, Lemma 2.2.5], shows that the 2-groupoid of pairs (G,μ), where $\mathrm{G} \to \mathrm{Spec}(\mathrm{F})$ is a reductive group and μ is an étale metaplectic cover of G, is the filtered colimit over nonempty k-finite subschemes $\mathrm{D} \subset \mathrm{X}$ of the 2-groupoid of pairs (G_1,μ_1), where $\mathrm{G}_1 \to \mathrm{X}$ is a smooth affine group scheme, reductive over $\mathrm{X}\backslash\mathrm{D}$ and parahoric along D, and μ_1 is an étale metaplectic cover of the restriction $\mathrm{G}_{1,\mathrm{X}\backslash\mathrm{D}}$.

We thus obtain (0.3) as the filtered colimit of (4.24), applied to each (G_1,μ_1) as above, over nonempty k-finite subschemes $\mathrm{D} \subset \mathrm{X}$.

Remark 4.4.9. In contrast to the non-metaplectic context, the generic version (0.3) does not allow us to state its compatibility with Satake isomorphism in a canonical manner.

Namely, for μ to be "unramified" at a closed point $x \in \mathrm{X}$ involves the *datum* of an extension across x which is generally not unique (Remark 4.1.5). Distinct choices of extensions give rise to distinct sections of $\widetilde{\mathrm{G}}_x \to \mathrm{G}(\mathrm{F}_x)$ over $\mathrm{G}(\mathscr{O}_x)$, and correspondingly distinct splittings of the L-group ${}^{\mathrm{L}}\mathrm{H}_{\mathrm{F},\vartheta} \to \mathrm{Gal}(\bar{\mathrm{F}}/\mathrm{F})$ over the inertia subgroup $\mathrm{I}_x \subset \mathrm{Gal}(\bar{\mathrm{F}}_x/\mathrm{F}_x)$.

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